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A LOCAL VARIATIONAL RELATION AND APPLICATIONS*

BY

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ABSTRACT

In [BGH] the authors show that for a given topological dynamical system (X,T) and an open cover $\mathcal{U}$ there is an invariant measure μ such that $\inf h_\mu(T,\mathbb{P}) \geq h_{\text{top}}(T,\mathcal{U})$ where infimum is taken over all partitions finer than $\mathcal{U}$. We prove in this paper that if μ is an invariant measure and $h_\mu(T,\mathbb{P}) > 0$ for each $\mathbb{P}$ finer than $\mathcal{U}$, then $\inf h_\mu(T,\mathbb{P}) > 0$ and $h_{\text{top}}(T,\mathcal{U}) > 0$. The results are applied to study the topological analogue of the Kolmogorov system in ergodic theory, namely uniform positive entropy (u.p.e.) of order n ($n \geq 2$) or u.p.e. of all orders. We show that for each $n \geq 2$ the set of all topological entropy n-tuples is the union of the set of entropy n-tuples for an invariant measure over all invariant measures. Characterizations of positive entropy, u.p.e. of order n and u.p.e. of all orders are obtained.

We could answer several open questions concerning the nature of u.p.e. and c.p.e.. Particularly, we show that u.p.e. of order n does not imply u.p.e. of order $n+1$ for each $n \geq 2$. Applying the methods and results obtained in the paper, we show that u.p.e. (of order 2) system is weakly disjoint from all transitive systems, and the product of u.p.e. of order n (resp. of all orders) systems is again u.p.e. of order n (resp. of all orders).

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1. Introduction

Ergodic theory and topological dynamics exhibit a remarkable parallelism. For example, ergodicity, weak mixing and mixing in ergodic theory can be translated as transitivity, topological weak mixing and topological mixing. The K-system in ergodic theory is an important class and it completely differs from zero entropy systems. It is well known that a measurable system is K if and only if it has completely positive entropy (each non-trivial factor has positive entropy) if and only if every partition by two non-trivial elements has positive entropy if and only if every partition by finite non-trivial elements has positive entropy. Using the first two conditions Blanchard [B1] introduces the notion of c.p.e. and u.p.e. in topological dynamics as an analogue of the K-system in ergodic theory, and shows that u.p.e. implies weak mixing and c.p.e. implies the existence of an invariant measure with full support. He then naturally defines the notion of entropy pairs and uses it to show that a u.p.e. system is disjoint from all minimal zero entropy systems [B2], and then with Lacroix [BL] to show that there is a maximal zero entropy factor associated to each topological dynamical system, an analogous notion of the Pinsker factor in ergodic theory. Later on, Glasner and Weiss [GW1] show that if a topological dynamical system admits a K-measure with full support then it has u.p.e., and in [B-R] the authors are able to define entropy pairs for a measure and show that for each invariant measure the set of entropy pairs for a measure is contained in the set of entropy pairs. Blanchard, Glasner and Host [BGH] show that the converse of [B-R] is also valid. Characterizing the set of entropy pairs for an ergodic measure as the support of some measure, Glasner [G] shows that the product of two u.p.e. systems has u.p.e. The topic on the relative notion of c.p.e. and u.p.e. can be found in [GW2]. Further research concerning the above results can be found in [KS], [LS]. Following the idea of entropy pairs one can also define complexity pairs [BHM] and [HY], sequence entropy pairs [HLSY] and sequence entropy pairs for a measure [HMY].

Despite great achievements, there are still many problems which remain open. The most vexing ones (as we understand) are: if u.p.e. of all orders is equivalent to u.p.e., how to define the entropy n-tuple for an invariant measure when $n > 2$ (the previous definition for $n = 2$ is not valid for $n > 2$)? We will give complete answers to these questions in this paper.

Looking back at the results on the relation between the two kinds of entropy pairs one finds that the local relation linking topological entropy for an open cover and metric entropy for a partition plays the central role. In [BGH] the authors show that for a given topological dynamical system (X, T) and an open cover $\mathcal{U}$ there is an invariant measure μ such that $\inf h_\mu(T, \mathbb{P}) \geq h_{\text{top}}(T, \mathcal{U})$ where infimum is taken over all partitions finer than $\mathcal{U}$. We prove in this paper that if μ is an invariant measure and $h_\mu(T, \mathbb{P}) > 0$ for each partition finer than $\mathcal{U}$, then $\inf h_\mu(T, \mathbb{P}) > 0$ and $h_{\text{top}}(T, \mathcal{U}) > 0$. The two local variational relations are applied to study the topological analogue of the Kolmogorov system in ergodic theory, namely uniform positive entropy (u.p.e.) of order n ($n \geq 2$) or u.p.e. of all orders. Localizing the notion of u.p.e. of order n, we define topological entropy n-tuples (n-topo), and entropy n-tuples for an invariant measure (n-meas). We show that for each $n \geq 2$ the set of all n-topo is the union of the set of n-meas over all invariant measures. It turns out that if a topological dynamical system admits an invariant K-measure with full support, then it has u.p.e. of all orders. The above results generalize previous ones by many authors. Characterizations of u.p.e. of order n and u.p.e. of all orders connecting with a topologically non-trivial measurable partition (Theorem 6.7) or interpolating set of positive density (Theorem 8.3) or Property P_n (Theorem 7.4) or hyperspace (Theorem 8.4) are given (they are new even for $n = 2$). It turns out that u.p.e. of order 2 is close to Property P and u.p.e. of all orders is somehow related to the weak specification property. Moreover, a characterization of positive entropy via interpolating sets is obtained.

We could answer several open questions concerning the nature of u.p.e. and c.p.e. Namely, we show that u.p.e. of order n does not imply u.p.e. of order $n+1$ for each $n \geq 2$ (answering a question by Host, which is restated in [GW2]), that there is a transitive diagonal system which does not have u.p.e. (of order 2) [B2, Question 1], and that there is a u.p.e. (of order 2) system having no ergodic measure with full support [B1, Question 2]. Applying the methods and results obtained in the paper, we show that a u.p.e. (of order 2) system is weakly disjoint from all transitive systems, and that the product of u.p.e. of order n (of all orders) systems is again u.p.e. of order n (of all orders).

For the philosophical question, what is the best analogue of the K-system in the topological setting, we think that u.p.e. of all orders is a good candidate: Firstly, by the definition for each finite open cover with non-dense elements it has positive entropy; secondly, u.p.e. of order 2 does not imply u.p.e. of order 3; and finally, a system has u.p.e. of all orders if and only if there is invariant measure μ such that for each topologically non-trivial finite partition $\mathbb{P}$, $h_\mu(T, \mathbb{P}) > 0$.

This paper is organized as follows. In section 2, we introduce some necessary notions and in section 3 we prove that a dynamical system admitting an invariant K-measure with full support has in fact u.p.e. of all orders. In section 4 we define entropy n-tuples for a measure and show that they have lifting property and the set of entropy n-tuples for a measure is the support of some measure on the product space. Note that we heavily use Rohklin's result on the structure of Lebesgue spaces [R]. In section 5 we prove a theorem on the interrelation of the entropy of measurable covers and partitions; together with [BGH] we deepen our understanding of the variational principle. We remark that the previous proof in [B-R] is valid only for $n = 2$. Section 6 deals with the relation of n-topo and n-meas. In section 7 and section 8 we give other characterizations of u.p.e. of order n and u.p.e. of all orders, and use the results to prove that u.p.e. of order 2 system is weakly disjoint from all transitive systems. In section 9 we give the examples.

We would like to thank the referee of the paper for the careful reading. The main results of the paper were obtained several years ago. While modifying it, some ideas of the paper have been applied to obtain some other results in [HMY], [HMPY] and [HSY]. Particularly, it is shown in [HSY] that a minimal topological system is strongly mixing. It is worth mentioning that just recently Glasner and Weiss [GW3] showed that $\inf h_\mu(T,\mathbb{P}) \le h_{\rm top}(T,\mathcal{U})$ for each invariant measure μ.

2. Preliminary

Let $(Y,\mathcal{D},\nu,T)$ be a measure-theoretic dynamical system (MDS, for short) and P_ν be its Pinsker σ-algebra. For a finite measurable partition $\mathbb{P}$, let $\mathbb{P}^- = \bigvee_{i=1}^{+\infty} T^{-i}\mathbb{P}$, and $H_\nu(\mathbb{P}|\mathcal{A})$ be the conditional entropy of $\mathbb{P}$ with respect to a σ-algebra $\mathcal{A}$. As usual, $\mathbb{P}\vee T^{-1}\mathbb{P}\vee\cdots\vee T^{-(n-1)}\mathbb{P}$ is denoted by $\mathbb{P}_0^{n-1}(T)$ or simply $\mathbb{P}_0^{n-1}$.

Recall that

$$h_\nu(T,\mathbb{P}) = \lim_{n\to+\infty}\frac{1}{n}H_\nu(\mathbb{P}_0^{n-1}) = H_\nu(\mathbb{P}|\mathbb{P}^-) = H_\nu(\mathbb{P}|\mathbb{P}^-\vee P_\nu)$$

and

$$\lim_{k\to+\infty} T^{-k}\mathbb{P}^-\vee P_\nu = P_\nu.$$

A **topological dynamical system** (TDS, for short) is a pair (X,T), where X is a compact metric space and T is a homeomorphism of X to itself. Given a finite cover $\mathcal{U}$ of X one defines the **combinatorial entropy** of $\mathcal{U}$ by the usual formula

$$h_c(T,\mathcal{U}) = \lim_{n\to+\infty}\frac{1}{n}H(\mathcal{U}\vee T^{-1}(\mathcal{U})\vee\cdots\vee T^{-(n-1)}(\mathcal{U}));$$

the limit exists since $H(\mathcal{U}) = \log\inf \#(\mathcal{U}')$ is sub-additive, where the infimum is taken over all subcovers $\mathcal{U}'$ of $\mathcal{U}$ and $\#$ denotes the cardinality. Note that $h_c(T,\mathcal{U})$ coincides with $h_{\text{top}}(T,\mathcal{U})$ when $\mathcal{U}$ is a finite **open** cover of X. By $\mathcal{U} \preceq \mathcal{V}$, we mean that the cover $\mathcal{U}$ is coarser than the cover $\mathcal{V}$ and the same notation will be used for partitions as well. When $\mathcal{U} \preceq \mathcal{V}$, we have $h_c(T,\mathcal{U}) \le h_c(T,\mathcal{V})$. Since a finite partition $\mathbb{P}$ of X is also a cover, it has also combinatorial entropy. In this case one has

$$h_c(T,\mathbb{P}) = \lim_{n\to+\infty} \frac{1}{n} \log \left| \bigvee_{i=0}^{n-1} T^{-i}\mathbb{P} \right|,$$

where $|\bigvee_{i=0}^{n-1} T^{-i}\mathbb{P}|$ is the number of non-empty elements in $\bigvee_{i=0}^{n-1} T^{-i}\mathbb{P}$. Hence for a Borel invariant probability measure μ, $h_\mu(T,\mathbb{P}) \le h_c(T,\mathbb{P})$.

The notion of topological entropy pairs is introduced in [B2]. Here we have (see also [GW1])

Definition 2.1: Let (X,T) be a TDS and $X^{(n)} = X \times \cdots \times X$ (n times) with $n \ge 2$. An n-tuple $(x_i)_1^n \in X^{(n)}$ is a **topological entropy n-tuple**(n-topo, for short) if at least two of the points $\{x_i\}_{i=1}^n$ are different and if whenever U_j are closed mutually disjoint neighborhoods of distinct points x_j, the open cover $\mathcal{U} = \{U_j^c\colon 1 \le j \le n\}$ has positive topological entropy, i.e. $h_{\text{top}}(T,\mathcal{U}) > 0$.

Let $n \ge 2$, $X^{(n)} = X \times \cdots \times X$ (n times) and $(x_i)_{i=1}^n \in X^{(n)}$. Let $\mathcal{U} = \{U_1, U_2, \ldots, U_n\}$ be a finite cover of X. We call $\mathcal{U}$ an **admissible cover** with respect to $(x_i)_{i=1}^n$ if for each U_i $(1 \le i \le k)$ there exists x_{j_i} $(1 \le j_i \le n)$ such that x_{j_i} is not in the closure of U_i.

Remark 2.2: It is easy to see that an n-tuple $(x_i)_1^n \in X^{(n)}$ is a topological entropy n-tuple if and only if at least two of the points $\{x_i\}_{i=1}^n$ are different and for any admissible finite open cover $\mathcal{U}$ with respect to $(x_i)_1^n \in X^{(n)}$ one has $h_{\text{top}}(T,\mathcal{U}) > 0$.

Definition 2.3: Let (X,T) be a TDS. (X,T) has **uniform positive entropy of order** n (u.p.e. of order n, for short), if for every point $(x_i)_1^n \in X^{(n)}$ not on the diagonal $\Delta_n(X) = \{(x)_1^n\colon x \in X\}$ is n-topo. We say (X,T) has u.p.e. of all orders or topo-K if it has u.p.e. of order n for every $n \ge 2$.

Clearly, a topological entropy 2-tuple is just a topological entropy pair and u.p.e. of order 2 is just u.p.e. defined in [B1].

Remark 2.3: It is easy to see that

(1) (X,T) has u.p.e. of order n if and only if any cover of X by n non-dense open sets has positive topological entropy.
(2) (X,T) is u.p.e. of all orders if and only if any cover of X by finitely many non-dense open sets has positive topological entropy.

For $n \geq 2$ denote by $E_n(X,T)$ the set of all n-topo, by $E'_n(X,T)$ its closure. The proofs of the following results are similar to that of the corresponding results in [B2].

PROPOSITION 2.4: *Let (X,T) be a TDS.*

(a) *If $\mathcal{U} = \{U_1, \dots, U_n\}$ is an open cover of X with $h_{\mathrm{top}}(T,\mathcal{U}) > 0$, then there are n points $x_i \in U_i^c$ for $1 \leq i \leq n$ such that $(x_i)_1^n$ is n-topo.*
(b) *$E'_n(X,T)$ is a nonempty closed $T^{(n)}$-invariant subset of $X^{(n)}$ containing only n-topo and points of $\Delta_n(X)$.*
(c) *Let $\pi\colon (Y,S) \longrightarrow (X,T)$ be a factor map of TDS.*
    (1) *If $(x_i)_1^n \in E_n(X,T)$, then there exist $y_i \in Y$, $1 \leq i \leq n$, such that $\pi(y_i) = x_i$ and $(y_i)_1^n \in E_n(Y,S)$.*
    (2) *Conversely, if $(y_i)_1^n \in E_n(Y,S)$ and $(\pi(y_i))_1^n \notin \Delta_n(Y)$, then $(\pi(y_i))_1^n$ belongs to $E_n(X,T)$.*
(d) *Suppose W is a closed T-invariant subset of (X,T). Then if $(x_i)_1^n$ is n-topo of $(W, T|_W)$, it is also n-topo of (X,T).*

3. K-measure with full support implies u.p.e. of all orders

In this section we shall show that if (X,T) is a TDS and admits an invariant K-measure with full support, then it has u.p.e. of all orders. Under the same assumption Glasner and Weiss [GW1] show that it has u.p.e. of order 2, and here by using a combinatorial result we avoid a complicated calculation and can prove that in fact it has u.p.e. of all orders. The result in this section indeed can be obtained from results in the later sections and we include the proof here to illustrate the basic ideas (in fact it is the starting point of the research). To do so, we need some lemmas. The first one is simple.

LEMMA 3.1: *Let X be a compact metric space and μ be a non-atomic probability measure on the Borel σ-algebra $\mathcal{B}(X)$ of X. If $B \in \mathcal{B}(X)$ with $\mu(B) \geq r > 0$, then for any $0 \leq \theta \leq r$ there exists a Borel set B_θ such that $B_\theta \subset B$ and $\mu(B_\theta) = \theta$.*

The second one is well known; see, for example, [BGH].

LEMMA 3.2: *Let $(Y, \mathcal{D}, \nu, T)$ be a MDS and P_ν be its Pinsker σ-algebra. Then for any finite measurable partition $\mathbb{P}$ of Y, $\lim_{n\to+\infty} h_\nu(T^n, \mathbb{P}) = H_\nu(\mathbb{P}|P_\nu)$.*

Let (X, T) be a TDS and C be a subset of X. If $\alpha = \{A_1, A_2, \ldots, A_n\}$ is a finite cover of C and $k \le n$, then

$$k\alpha = \{A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_k} : 1 \le i_1 < i_2 < \cdots < i_k \le n\}$$

is again a finite cover of C. Let $N(\alpha, C)$ be the minimum number of sets in any finite subcover of α covering C. Note that $\alpha \vee T^{-1}\alpha \vee \cdots \vee T^{-(n-1)}\alpha$ is denoted by $\alpha_0^{n-1}(T)$ or simply α_0^{n-1}. We have (see [X])

LEMMA 3.3:

(1) $N(\alpha, C) \le kN(k\alpha, C)$.

(2) $T^{-m}(k\alpha) = kT^{-m}(\alpha)$, *for* $m \in \mathbb{Z}$.

(3) *If* $\alpha_0, \ldots, \alpha_{l-1}$ *are covers of* C, *then* $k\alpha_0 \vee k\alpha_1 \vee \cdots \vee k\alpha_{l-1}$ *is a subcover of* $k^l(\alpha_0 \vee \alpha_1 \vee \cdots \vee \alpha_{l-1})$, *where* $k \le \min_{0\le i\le l-1} \#(\alpha_i)$, *and*

$$\begin{aligned} k^l N(k\alpha_0 \vee k\alpha_1 \vee \cdots \vee k\alpha_{l-1}, C) &\ge k^l N(k^l(\alpha_0 \vee \alpha_1 \vee \cdots \vee \alpha_{l-1}), C) \\ &\ge N(\alpha_0 \vee \alpha_1 \vee \cdots \vee \alpha_{l-1}, C). \end{aligned}$$

(4) *If* α *is a finite cover of* X *and* $k \le \#(\alpha)$, *then* $h_c(T, k\alpha) \ge h_c(T, \alpha) - \log k$.

Proof: (1)–(3) are easy to prove, and (4) can be proved using (1)–(3). ∎

With the above preparation we can now show the main result of the section. Note that we use $M(X, T)$ to denote the set of all Borel invariant probability measures under T.

THEOREM 3.4: *Let (X, T) be a TDS. Suppose there exists $\mu \in M(X, T)$ which is a K-measure with full support; then (X, T) has u.p.e. of all orders.*

Proof: Since (X, T, μ) is a measure-theoretic K-system, μ is non-atomic. For any $n \ge 2$ and n different points $x_1, x_2, \ldots, x_n$, let U_i be closed mutually disjoint neighborhoods of x_i. Put $r = \min_{1\le i\le n}\{\mu(U_i)\}$. Since $\mathrm{Supp}(\mu) = X$, we have $r > 0$. Choose $k \in \mathbb{N}$ such that $0 < 1/k < r$ and $k \ge n+1$. By Lemma 3.1, there exists a measurable partition $\mathbb{P} = \{B_1, B_2, \ldots, B_k\}$ with $B_i \subset U_i, i = 1, 2, \ldots, n$ and $\mu(B_j) = 1/k$, for $j = 1, 2, \ldots, k$.

Let P_μ be the Pinsker σ-algebra of (X, T, μ). As μ is a K-measure, $P_\mu = \{X, \emptyset\}$. Since

$$\lim_{n\to+\infty} h_\mu(T^n, \mathbb{P}) = H_\mu(\mathbb{P}|P_\mu) = H_\mu(\mathbb{P}) = \log k$$

(by Lemma 3.2 and $P_\mu = \{X, \emptyset\}$), there exists $l \in \mathbb{N}$ such that $h_\mu(T^l, \mathbb{P}) > \log(k-1)$.

Let $\mathcal{U} = \{U_1^c, U_2^c, \ldots, U_n^c\}$ and $\mathcal{V} = \{B_1^c, B_2^c, \ldots, B_k^c\}$. It is easy to see that $\mathcal{V} \preceq \mathcal{U}$ and $\mathcal{V} = \{\bigcup_{i\neq 1} B_i, \bigcup_{i\neq 2} B_i, \ldots, \bigcup_{i\neq k} B_i\} = (k-1)\mathbb{P}$.

Therefore,

$$\begin{aligned} h_{\rm top}(T,\mathcal{U}) &\geq \frac{1}{l} h_{\rm top}(T^l,\mathcal{U}) = \frac{1}{l} h_c(T^l,\mathcal{U}) \geq \frac{1}{l} h_c(T^l,\mathcal{V}) \\ &\geq \frac{1}{l}(h_c(T^l,\mathbb{P}) - \log(k-1)) \quad \text{(by Lemma 3.3 (4))} \\ &\geq \frac{1}{l}(h_\mu(T^l,\mathbb{P}) - \log\,(k-1)) > 0. \end{aligned}$$

Hence T has u.p.e. of all orders. ∎

Using Theorem 3.4 and Proposition 2.4, and following the proof of the corresponding results in [GW1], we have

THEOREM 3.5: *Given an arbitrary ergodic MDS $(Y, \mathcal{D}, \nu, T)$ with positive entropy, there exists a strictly ergodic u.p.e. of all orders TDS (X, T) with an invariant measure μ such that the systems $(Y, \mathcal{D}, \nu, T)$ and $(X, \mathcal{B}, \mu, T)$ are measure-theoretically isomorphic.*

4. Entropy n-tuples for an invariant measure

Let (X,T) be a TDS, $\mu \in M(X,T)$ and $\mathcal{B} = \mathcal{B}(X)$ be the Borel σ-algebra of X. In [B-R] the authors introduce the notion of entropy pairs for a measure and the notion cannot be directly generalized for n-tuples when $n > 2$. In this section we will give a definition of entropy n-tuples for a measure which is the same as the previous notion when $n = 2$. Then we show that the set of entropy n-tuples for a measure is the support of the disintegration of the measure over the Pinsker factor (see Glasner [G]). Finally, we show that entropy n-tuples for a measure have the lifting property.

Let $\mathcal{B}_\mu$ be the completion of $\mathcal{B}$ under μ. $A \subset X$ is a **μ-set** if $A \in \mathcal{B}_\mu$ and $A \subset X$ is a **Borel set** if $A \in \mathcal{B}$. Now we define entropy n-tuples for μ.

Definition 4.1: Let $n \geq 2$ and $(x_i)_1^n \in X^{(n)} \setminus \Delta_n(X)$. By an **admissible partition** $\mathbb{P}$ with respect to $(x_i)_1^n$ we mean

1. $\mathbb{P}$ is a finite Borel partition of X and
2. if $\mathbb{P} = \{A_1, \ldots, A_k\}$, then for each A_i $(1 \leq i \leq k)$ there exists x_{j_i} $(1 \leq j_i \leq n)$ such that $x_{j_i} \notin cl(A_i)$.

$(x_i)_1^n$ is an entropy n-tuple for μ, if for any admissible partition $\mathbb{P}$ with respect to $(x_i)_1^n$ we have $h_\mu(T,\mathbb{P}) > 0$.

Remark 4.2: In the definition of an admissible partition, we may replace (1) by (1)′: $\mathbb{P}$ is a finite μ-set partition of X.

Denote by $E_n^\mu(X,T)$ the set of all entropy n-tuples for μ ($n \geq 2$). In the following, we investigate the structure of $E_n^\mu(X,T)$.

Let $(Y,\mathcal{D},\nu,T)$ be a MDS and P_ν be its Pinsker σ-algebra. Define a measure $\lambda_n(\nu)$ on $(Y^{(n)},\mathcal{D}^{(n)},T^{(n)})$ by letting

$$\lambda_n(\nu)\left(\prod_{i=1}^n A_i\right) = \int_X \prod_{i=1}^n \mathbb{E}(1_{A_i}|P_\nu)d\nu,$$

where $\mathcal{D}^{(n)} = \mathcal{D} \times \cdots \times \mathcal{D}$ (n times), $T^{(n)} = T \times \cdots \times T$ (n times) and $A_i \in \mathcal{D}$, $i = 1,2,\ldots,n$. To get a characterization of $E_n^\mu(X,T)$ we need

LEMMA 4.3: *Let $(Y,\mathcal{D},\nu,T)$ be a MDS and $\mathcal{U} = \{U_1,U_2,\ldots,U_n\}$ be a measurable cover of X. Then $\lambda_n(\nu)(\prod_{i=1}^n U_i^c) > 0$ if and only if for any finite measurable (or n-set) partition $\mathbb{P}$, finer than $\mathcal{U}$ as a cover, one has $h_\nu(T,\mathbb{P}) > 0$.*

Proof: Assume that for any finite measurable (or n-set) partition $\mathbb{P}$, finer than $\mathcal{U}$ as a cover, one has $h_\nu(T,\mathbb{P}) > 0$ and $\lambda_n(\nu)(\prod_{i=1}^n U_i^c) = 0$.

Let $C_i = \{x \in X : \mathbb{E}(1_{U_i^c}|P_\nu)(x) > 0\} \in P_\nu$. As

$$0 = \int_{X\setminus C_i} \mathbb{E}(1_{U_i^c}|P_\nu)(x)d\nu = \nu(U_i^c \cap (X \setminus C_i)),$$

we have $\nu(U_i^c \setminus C_i) = 0$, $1 \leq i \leq n$. Put $D_i = C_i \cup (U_i^c \setminus C_i)$; then $D_i \in P_\nu$ and $D_i^c \subset U_i$, $1 \leq i \leq n$. For any $s = (s(1),\ldots,s(n)) \in \{0,1\}^n$, let $D_s = \bigcap_{i=1}^n D_i(s(i))$, where $D_i(0) = D_i$ and $D_i(1) = X \setminus D_i$. Set $D_0^j = (\bigcap_{i=1}^n D_i) \cap (U_j \setminus \bigcup_{k=1}^{j-1} U_k)$ for $j = 1,2,\ldots,n$.

Consider a measurable partition

$$\mathbb{P} = \{D_s \colon s \in \{0,1\}^n \quad \text{and} \quad s \neq (0,0,\ldots,0)\} \cup \{D_0^1, D_0^2, \ldots, D_0^n\}.$$

For any $s \in \{0,1\}^n$ with $s \neq (0,0,\cdots,0)$, one has $s(i) = 1$ for some $1 \leq i \leq n$. Then $D_s \subset D_i^c \subset U_i$ and clearly, $D_0^j \subset U_j$, $j = 1,2,\ldots,n$. Thus $\mathbb{P}$ is finer than $\mathcal{U}$, and $h_\nu(T,\mathbb{P}) > 0$.

On the other hand, since $\lambda_n(\nu)(\prod_{i=1}^n U_i^c) = 0$, it is easy to show that $\nu(\bigcap_{i=1}^n D_i) = \nu(\bigcap_{i=1}^n C_i) = 0$. Thus one has $D_0^1, D_0^2, \ldots, D_0^n \in P_\nu$. It is clear that $D_s \in P_\nu$ for $s \in \{0,1\}^n$, since $D_1, D_2, \ldots, D_n \in P_\nu$. As each element of $\mathbb{P}$ belongs to P_ν, one gets $h_\nu(T,\mathbb{P}) = 0$, a contradiction.

Now assume $\lambda_n(\mu)(\Pi_{i=1}^n U_i^c) > 0$. For any finite measurable partition $\mathbb{P}$ finer than $\mathcal{U}$ as a cover, without loss of generality, we assume that

$$\mathbb{P} = \{A_1, A_2, \ldots, A_n\}$$

with $A_i \subset U_i$, $i = 1, 2, \ldots, n$.

Note that

$$\begin{aligned}\int_X \prod_{i=1}^n \mathbb{E}(1_{X\setminus A_i}|P_\nu)(x)d\nu(x) &\geq \int_X \prod_{i=1}^n \mathbb{E}(1_{U_i^c}|P_\nu)(x)d\nu(x) \\ &= \lambda_n(\nu)\Big(\prod_{i=1}^n U_i^c\Big) > 0.\end{aligned}$$

Therefore, $A_j \notin P_\nu$ for every $1 \leq j \leq n$. This implies $h_\nu(T,\mathbb{P}) > 0$. ∎

Now we show a characterization of $E_n^\mu(X,T)$ for any $n \geq 2$ and we remark that the case $n = 2$ is proved in [G].

THEOREM 4.4: *Let (X,T) be a TDS and $\mu \in M(X,T)$. Then $E_n^\mu(X,T) = \mathrm{Supp}(\lambda_n(\mu)) \setminus \Delta_n$, where $n \geq 2$.*

Proof: Let $(x_i)_1^n \in E_n^\mu(X,T)$. To show $(x_i)_1^n \in \mathrm{Supp}(\lambda_n(\mu)) \setminus \Delta_n$, it remains to prove that for any neighborhood U_i of x_i, $\lambda_n(\mu)(\prod_{i=1}^n U_i) > 0$.

Set $\mathcal{U} = \{U_1^c, U_2^c, \ldots, U_n^c\}$. Without loss of generality, we assume that $\mathcal{U}$ is a measurable cover of X (if needed, replace U_i by a smaller neighborhood). It is clear that for any finite measurable partition $\mathbb{P}$, finer than $\mathcal{U}$ as a cover, $\mathbb{P}$ is an admissible partition with respect to $(x_i)_1^n$. Therefore $h_\mu(T,\mathbb{P}) > 0$. By Lemma 4.3, $\lambda_n(\mu)(\prod_{i=1}^n U_i) > 0$.

Now assume $(x_i)_1^n \in \mathrm{Supp}(\lambda_n(\mu)) \setminus \Delta_n$. We will show that $h_\mu(T,\mathbb{P}) > 0$ for any admissible partition $\mathbb{P} = \{A_1, A_2, \ldots, A_k\}$ with respect to $(x_i)_1^n$.

Since $\mathbb{P}$ is an admissible partition with respect to $(x_i)_1^n$, there exists neighborhood U_i of x_i such that for each $i \in \{1,2,\ldots,k\}$ we find $j_i \in \{1,2,\ldots,n\}$ with $A_i \subset U_{j_i}^c$. That is, $\mathbb{P}$ is finer than $\mathcal{U} = \{U_1^c, U_2^c, \ldots, U_n^c\}$ as a cover. As $\lambda_n(\mu)(\prod_{i=1}^n U_i) > 0$, one has $h_\mu(T,\mathbb{P}) > 0$ by Lemma 4.3. ∎

From now on we proceed to show that the lifting property is valid for entropy tuples for a measure (Theorem 4.10). To do so we need

LEMMA 4.5: *Let α, β be finite measurable partitions of a MDS $(Y, \mathcal{D}, \nu, T)$. Then $h_\nu(T, \alpha) \le h_\nu(T, \beta) + H_\nu(\alpha|\beta \vee P_\nu)$.*

Proof: Let $a_n = H_\nu(\alpha_0^{n-1}|\beta_0^{n-1} \vee P_\nu)$. First we show that a_n is sub-additive. In fact

$$\begin{aligned}
a_{n+m} =& H_\nu((\alpha \vee \beta)_0^{n+m-1}|P_\nu) - H_\nu(\beta_0^{n+m-1}|P_\nu) \\
=& H_\nu((\alpha \vee \beta)_0^{n-1} \vee T^{-n}(\alpha \vee \beta)_0^{m-1}|P_\nu) - H_\nu(\beta_0^{n-1} \vee T^{-n}\beta_0^{m-1}|P_\nu) \\
=& H_\nu(T^{-n}(\alpha \vee \beta)_0^{m-1}|P_\nu) + H_\nu((\alpha \vee \beta)_0^{n-1}|T^{-n}(\alpha \vee \beta)_0^{m-1} \vee P_\nu) \\
& - H_\nu(\beta_0^{n-1} \vee T^{-n}\beta_0^{m-1}|P_\nu) \\
\le& H_\nu(T^{-n}(\alpha \vee \beta)_0^{m-1}|P_\nu) + H_\nu((\alpha \vee \beta)_0^{n-1}|T^{-n}\beta_0^{m-1} \vee P_\nu) \\
& - H_\nu(\beta_0^{n-1} \vee T^{-n}\beta_0^{m-1}|P_\nu) \\
=& H_\nu(T^{-n}(\alpha \vee \beta)_0^{m-1}|P_\nu) - H_\nu(T^{-n}\beta_0^{m-1}|P_\nu) \\
& + H_\nu(\alpha_0^{n-1} \vee \beta_0^{n-1} \vee T^{-n}\beta_0^{m-1}|P_\nu) - H_\nu(\beta_0^{n-1} \vee T^{-n}\beta_0^{m-1}|P_\nu) \\
=& a_m + H_\nu(\alpha_0^{n-1}|\beta_0^{n-1} \vee T^{-n}\beta_0^{m-1} \vee P_\nu) \le a_m + a_n.
\end{aligned}$$

Since a_n is sub-additive, $\lim_{n \to +\infty}(a_n/n) = \inf_{n \ge 1}(a_n/n)$. Therefore,

$$\begin{aligned}
h_\nu(T, \alpha) - h_\nu(T, \beta) &\le h_\nu(T, \alpha \vee \beta) - h_\nu(T, \beta) \\
&= \lim_{n \to +\infty} \frac{H_\nu((\alpha \vee \beta)_0^{n-1}|P_\nu) - H_\nu(\beta_0^{n-1}|P_\nu)}{n} \\
&= \lim_{n \to +\infty} \frac{a_n}{n} = \inf_{n \ge 1} \frac{a_n}{n} \le H_\nu(\alpha|\beta \vee P_\nu). \qquad \blacksquare
\end{aligned}$$

To simplify the notation we now introduce

Definition: Let $(Y, \mathcal{D}, \nu, T)$ be a MDS and $\mathcal{U} = \{U_1, U_2, \ldots, U_n\}$ be a measurable cover of Y. Set

$$h_\nu(T, \mathcal{U}) := \inf\{h_\nu(T, \mathbb{P}) : \mathbb{P} \text{ is a finite measurable partition finer than } \mathcal{U}\}.$$

It can be considered as the measure entropy of cover $\mathcal{U}$ with respect to ν.

The following theorem and Theorem 5.7 are crucial for our paper and will be useful in other settings as well.

THEOREM 4.6: *Let $(Y, \mathcal{D}, \nu, T)$ be a MDS and $\mathcal{U} = \{U_1, U_2, \ldots, U_n\}$ be a measurable cover of Y with $n \ge 2$. If $h_\nu(T, \mathbb{P}) > 0$ for every finite measurable (n-set) partition $\mathbb{P}$, finer than $\mathcal{U}$ as a cover, then $h_\nu(T, \mathcal{U}) > 0$.*

Proof: By Lemma 4.3, $\lambda_n(\nu)(\prod_{i=1}^n U_i^c) > 0$, i.e., $\int_X \prod_{i=1}^n \mathbb{E}(1_{U_i^c}|P_\nu)d\nu > 0$. Hence, there is a natural number M such that $\nu(D) > 0$, where

$$D = \{x \in X : \min_{1\le i\le n} \mathbb{E}(1_{U_i^c}|P_\nu)(x) \ge 1/M\}.$$

For any $s = (s(1), \ldots, s(n)) \in \{0,1\}^n$, set $A_s = \bigcap_{i=1}^n U_i(s(i))$, where $U_i(0) = U_i$ and $U_i(1) = X \setminus U_i$, and $\alpha = \{A_s : s \in \{0,1\}^n\}$. We have

CLAIM: *$H_\nu(\alpha|\beta \vee P_\nu) \le H_\nu(\alpha|P_\nu) - \frac{\nu(D)}{M}\log(\frac{n}{n-1})$, for any finite measurable partition β which is finer than $\mathcal{U}$ as a cover.*

Proof of Claim: Without loss of generality, assume $\beta = \{B_1, B_2, \ldots, B_n\}$ with $B_i \subset U_i$, $i = 1, 2, \ldots, n$.

Let $f(x) = \begin{cases} -x\log x & \text{if } x > 0, \\ 0 & \text{if } x = 0. \end{cases}$

Then

$$\begin{aligned} H_\nu(\alpha|\beta \vee P_\nu) &= H_\nu(\alpha \vee \beta|P_\nu) - H_\nu(\beta|P_\nu) \\ &= \int_X \sum_{s\in\{0,1\}^n} \sum_{i=1}^n \mathbb{E}(1_{B_i}|P_\nu) f\Big(\frac{\mathbb{E}(1_{A_s\cap B_i}|P_\nu)}{\mathbb{E}(1_{B_i}|P_\nu)}\Big)d\nu \\ &= \sum_{s\in\{0,1\}^n} \int_X \sum_{i,s(i)=0} \mathbb{E}(1_{B_i}|P_\nu) f\Big(\frac{\mathbb{E}(1_{A_s\cap B_i}|P_\nu)}{\mathbb{E}(1_{B_i}|P_\nu)}\Big)d\nu, \end{aligned} \tag{4.6.1}$$

where $i, s(i) = 0$ means "for all i with $s(i) = 0$". The last equality comes from the fact that for any $s \in \{0,1\}^n$ and $1 \le i \le n$ if $s(i) = 1$, then $A_s \cap B_i = \emptyset$ and the fact that

$$\frac{\mathbb{E}(1_{A_s\cap B_i}|P_\nu)}{\mathbb{E}(1_{B_i}|P_\nu)}(x) \equiv 0.$$

Put $c_s = \sum_{k,s(k)=0} \mathbb{E}(1_{B_k}|P_\nu)$. As f is concave,

$$\begin{aligned} (4.6.1) &\le \sum_{s\in\{0,1\}^n} \int_X c_s \cdot f\Big(\sum_{i,s(i)=0} \frac{\mathbb{E}(1_{B_i}|P_\nu)}{c_s}\frac{\mathbb{E}(1_{A_s\cap B_i}|P_\nu)}{\mathbb{E}(1_{B_i}|P_\nu)}\Big)d\nu \\ &= \sum_{s\in\{0,1\}^n} \int_X c_s \cdot f\Big(\frac{\sum_{i,s(i)=0}\mathbb{E}(1_{A_s\cap B_i}|P_\nu)}{c_s}\Big)d\nu \\ &= \sum_{s\in\{0,1\}^n} \int_X c_s \cdot f\Big(\frac{\mathbb{E}(1_{A_s}|P_\nu)}{c_s}\Big)d\nu \\ &= \sum_{s\in\{0,1\}^n} \Big(\int_X f(\mathbb{E}(1_{A_s}|P_\nu))d\nu - \int_X \mathbb{E}(1_{A_s}|P_\nu)\log(1/c_s)d\nu\Big) \\ &= H_\nu(\alpha|P_\nu) - \sum_{s\in\{0,1\}^n} \int_X \mathbb{E}(1_{A_s}|P_\nu)\log(1/c_s)d\nu. \end{aligned}$$

Note that if $s(i)=1$ $(1\le i\le n)$ then $\Sigma_{k,s(k)=0}\mathbb{E}(1_{B_k}|P_\nu)\le \mathbb{E}(1_{X\setminus B_i}|P_\nu)$, as β is a partition. Putting $b_i=\mathbb{E}(1_{X\setminus B_i}|P_\nu)$, $i=1,\ldots,n$, then we have

$$\begin{aligned}
&\sum_{s\in\{0,1\}^n}\int_X \mathbb{E}(1_{A_s}|P_\nu)\log\Big(\frac{1}{\sum_{k,s(k)=0}\mathbb{E}(1_{B_k}|P_\nu)}\Big)d\nu\\
&\ge \frac{1}{n}\sum_{i=1}^n\int_X\Big(\sum_{s,s(i)=1}\mathbb{E}(1_{A_s}|P_\nu)\Big)\log\frac{1}{b_i}d\nu\\
&=\frac{1}{n}\sum_{i=1}^n\int_X\mathbb{E}(1_{U_i^c}|P_\nu)\log\frac{1}{b_i}d\nu\ge\frac{1}{nM}\sum_{i=1}^n\int_D\log\frac{1}{b_i}d\nu\\
&=\frac{1}{nM}\int_D\log\frac{1}{\prod_{i=1}^n b_i}d\nu\ge\frac{1}{M}\int_D\log\frac{n}{\sum_{i=1}^n b_i}d\nu\\
&=\frac{\nu(D)}{M}\log\Big(\frac{n}{n-1}\Big),
\end{aligned}$$

as

$$\Big(\frac{b_1+\cdots+b_n}{n}\Big)^n\ge b_1\cdots b_n,$$

and

$$\sum_{i=1}^n b_i=\sum_{i=1}^n\sum_{j\ne i}\mathbb{E}(1_{B_j}|P_\nu)=(n-1)\sum_{i=1}^n\mathbb{E}(1_{B_i}|P_\nu)=n-1.$$

Hence

$$H_\nu(\alpha|\beta\vee P_\nu)\le H_\nu(\alpha|P_\nu)-\frac{\nu(D)}{M}\log\Big(\frac{n}{n-1}\Big).$$

This ends the proof of claim.

Put

$$\epsilon=\frac{\nu(D)}{M}\log\Big(\frac{n}{n-1}\Big)>0.$$

Since $\lim_{m\to+\infty}h_\nu(T^m,\alpha)=H_\nu(\alpha|P_\nu)$, there exists $l>0$ such that $h_\nu(T^l,\alpha)\ge H_\nu(\alpha|P_\nu)-\epsilon/2$. Now for any finite measurable partition β, finer than $\mathcal{U}$ as a cover, one has

$$h_\nu(T,\beta)\ge\frac{1}{l}h_\nu(T^l,\beta)\ge\frac{1}{l}(h_\nu(T^l,\alpha)-H_\nu(\alpha|\beta\vee P_\nu))\ge\frac{\epsilon}{2l},$$

by Lemma 4.5. This shows that $h_\nu(T,\mathcal{U})\ge\epsilon/2l$. ∎

An immediate consequence is

COROLLARY 4.7: *Let $(Y,\mathcal{D},\nu,T)$ be a MDS and $\mathcal{U}=\{U_1,U_2,\ldots,U_n\}$ be a measurable cover of Y. Then $h_\nu(T,\mathcal{U})>0$ if and only if $\lambda_n(\nu)(\prod_{i=1}^n U_i^c)>0$.*

Proof: It follows from Lemma 4.3 and Lemma 4.6. ∎

LEMMA 4.8: *Let (X,T) be a TDS, $\mu \in M(X,T)$ and $\mathcal{U}$ be a finite Borel cover of X. If $\mu = \int_\Omega \mu_\omega dm(\omega)$ is the ergodic decomposition of μ, then $h_\mu(T,\mathcal{U}) = \int_\Omega h_{\mu_\omega}(T,\mathcal{U})dm(\omega)$.*

Proof: Let $\mathcal{U} = \{U_1, U_2, \ldots, U_N\}$. Define

$$\mathcal{U}(P) = \{\beta\colon \beta = \{B_i \subset U_i\colon 1 \le i \le N\} \text{ is Borel partition of } X\}.$$

As X is a compact metric space, there exists a countable family $\{\mathbb{P}_i \in \mathcal{U}(P) : i \in \mathbb{N}\}$ such that it is $L^1(\nu)$-dense in $\mathcal{U}(P)$ for each probability measure ν on X. In fact, if $\{V_i\}$ is a base of X and $\mathcal{A}$ is the algebra generated by $\{V_i\} \cup \mathcal{U}$, then $\mathbb{P}_i$ can be taken such that its j-th element is in $\mathcal{A}$ and is contained in U_j, $1 \le j \le N$. Clearly, for each $\nu \in M(X,T)$, $h_\nu(T,\mathcal{U}) = \inf_{i\in\mathbb{N}} h_\nu(T,\mathbb{P}_i)$.

Let $\epsilon > 0$. As $h_{\mu_\omega}(T,\mathcal{U}) = \inf_{i\in\mathbb{N}} h_{\mu_\omega}(T,\mathbb{P}_i)$ for each $\mu_\omega \in \Omega$, there exists, therefore, a partition $\{\Omega_n\}_{n\in I\subset\mathbb{N}}$ of Ω $(\mathrm{mod}\, m)$ with $m(\Omega_n) > 0$ for every $n \in I$ such that $h_{\mu_\omega}(T,\mathbb{P}_n) < h_{\mu_\omega}(T,\mathcal{U}) + \epsilon$ if $\mu_\omega \in \Omega_n$.

For $n \in I$ write $t_n = m(\Omega_n)$ and $\mu_n = \frac{1}{t_n}\int_{\Omega_n} \mu_\omega dm(\omega)$. One has

$$\begin{aligned} h_{\mu_n}(T,\mathbb{P}_n) &= \frac{1}{t_n}\int_{\Omega_n} h_{\mu_\omega}(T,\mathbb{P}_n)dm(\omega) \\ &\le \frac{1}{t_n}\int_{\Omega_n} h_{\mu_\omega}(T,\mathcal{U})dm(\omega) + \epsilon. \end{aligned}$$

The measures $\{\mu_n\}$ are mutually singular, i.e., there exist Borel subsets $\{X_n\}_{n\in I}$ such that, for each $n, k \in I$, $\mu_n(X_n) = 1$ and $\mu_n(X_k) = 0$ for $k \ne n$. We can assume that $\{X_n\}_{n\in I}$ is a partition of X. Let $\mathbb{P}_n = \{B_i^n \subset U_i\colon 1 \le i \le N\}$ and $B_i = \bigcup_{n\in I}(X_n \cap B_i^n)$. Then $\mathbb{P} = \{B_1, B_2, \ldots, B_N\} \in \mathcal{U}(P)$ and $\mathbb{P} = \mathbb{P}_n$ $(\mathrm{mod}\, \mu_n)$ for each $n \in I$. We have

$$\begin{aligned} h_\mu(T,\mathbb{P}) &= \sum_n t_n h_{\mu_n}(T,\mathbb{P}) = \sum_n t_n h_{\mu_n}(T,\mathbb{P}_n) \\ &\le \int_\Omega h_{\mu_\omega}(T,\mathcal{U})dm(\omega) + \epsilon. \end{aligned}$$

Hence $h_\mu(T,\mathcal{U}) \le \int_\Omega h_{\mu_\omega}(T,\mathcal{U})dm(\omega) + \epsilon$ and thus

$$h_\mu(T,\mathcal{U}) \le \int_\Omega h_{\mu_\omega}(T,\mathcal{U})dm(\omega).$$

On the other hand,

$$\begin{aligned} h_\mu(T,\mathcal{U}) &= \inf_{i\in\mathbb{N}} h_\mu(T,\mathbb{P}_i) = \inf_{i\in\mathbb{N}}\int_\Omega h_{\mu_\omega}(T,\mathbb{P}_i)dm(\omega) \\ &\ge \int_\Omega h_{\mu_\omega}(T,\mathcal{U})dm(\omega). \end{aligned}$$

This shows that $h_\mu(T,\mathcal{U}) = \int_\Omega h_{\mu_\omega}(T,\mathcal{U})dm(\omega)$. ∎

With the help of Corollary 4.7 and Lemma 4.8 we now can show Theorem 4.9, which discloses the relation of entropy tuples for a measure and entropy tuples for ergodic measures in its ergodic decomposition, generalizing Theorem 4 of [BGH].

THEOREM 4.9: *Let (X,T) be a TDS, $\mu \in M(X,T)$ and $\mu = \int_\Omega \mu_\omega dm(\omega)$ be its ergodic decomposition.*

(i) *For m-almost every ω, $E_n^{\mu_\omega}(X,T) \subset E_n^\mu(X,T)$, where $n \geq 2$.*
(ii) *If $(x_i)_1^n \in E_n^\mu(X,T)$, then for every neighborhood V of $(x_i)_1^n$,*

$$m\{\omega: V \cap E_n^{\mu_\omega}(X,T) \neq \emptyset\} > 0.$$

Thus for an appropriate choice of Ω,

$$\mathrm{cl}(\bigcup\{E_n^{\mu_\omega}(X,T): \omega \in \Omega\}) \setminus \Delta_n(X) = E_n^\mu(X,T).$$

Proof: (i) Let U_i, $i = 1,2,\dots,n$ be open subsets of X with $\bigcap_{i=1}^n \mathrm{cl}(U_i) = \emptyset$ and $(\prod_{i=1}^n \mathrm{cl}(U_i)) \cap E_n^\mu(X,T) = \emptyset$. As $E_n^\mu(X,T) = \mathrm{Supp}\,\lambda_n(\mu) \setminus \Delta_n(X)$, $\lambda_n(\mu)(\prod_{i=1}^n \mathrm{cl}(U_i)) = 0$. By Corollary 4.7,

$$h_\mu(T,\mathcal{U}) = 0, \quad \text{where } \mathcal{U} = \{U_1^c, U_2^c, \dots, U_n^c\}.$$

As $\int_\Omega h_{\mu_\omega}(T,\mathcal{U})dm(\omega) = h_\mu(T,\mathcal{U}) = 0$, $h_{\mu_\omega}(T,\mathcal{U}) = 0$ for m-a.e. ω. By Corollary 4.7, $\lambda_n(\mu_\omega)(\prod_{i=1}^n(U_i)) = 0$ for m-a.e. ω. Hence $(\prod_{i=1}^n U_i) \cap E_n^{\mu_\omega}(X,T) = \emptyset$ for m-a.e. ω.

Since $E_n^\mu(X,T) \cup \Delta_n(X)$ is closed in $X^{(n)}$, its complement can be written as a countable union of sets of the form $\prod_{i=1}^n U_i$, where $U_i, i = 1,2,\dots,n$ are open subsets with $\bigcap_{i=1}^n \mathrm{cl}(U_i) = \emptyset$. By definition, $E_n^{\mu_\omega}(X,T) \cap \Delta_n(X) = \emptyset$ for all ω, and we conclude that for m-a.e. ω, $E_n^{\mu_\omega}(X,T) \cap (E_n^\mu(X,T))^c = \emptyset$.

(ii) Without loss of generality, we assume $V = \prod_{i=1}^n A_i$, with A_i a closed neighborhood of x_i and $\bigcap_{i=1}^n A_i = \emptyset$.

As $\lambda_n(\mu)(\prod_{i=1}^n A_i) > 0$, one has $h_\mu(T, \{A_1^c, A_2^c, \dots, A_n^c\}) > 0$. Since

$$\int_\Omega h_{\mu_\omega}(T, \{A_1^c, A_2^c, \dots, A_n^c\})dm(\omega) = h_\mu(T, \{A_1^c, A_2^c, \dots, A_n^c\}) > 0,$$

there exists $\Omega' \subset \Omega$ with $m(\Omega') > 0$ such that when $\omega \in \Omega'$,

$$h_{\mu_\omega}(T, \{A_1^c, A_2^c, \dots, A_n^c\}) > 0, \quad \text{i.e., } \lambda_n(\mu_\omega)(\prod_{i=1}^n A_i) > 0.$$

Clearly, $\omega \in \Omega'$, $(\Pi_{i=1}^n A_i) \cap E_n^{\mu_\omega}(X,T) \neq \emptyset$. This shows

$$m(\{w\colon V \cap E_n^{\mu_\omega}(X,T) \neq \emptyset\}) > 0. \qquad \blacksquare$$

The special case $n = 2$ of the following theorem is proved in [BGH] and now we can show

THEOREM 4.10: *Let $\pi\colon (X,T) \to (Y,S)$ be a factor map of TDS, $\mu \in M(X,T)$, and ν be its image under π.*

1. *For every $(x_i)_1^n \in E_n^\mu(X,T)$ let $\pi(x_i) = y_i$, $i = 1,2,\ldots,n$. If $(y_i)_1^n \notin \triangle_n(Y)$, then $(y_i)_1^n \in E_n^\nu(Y,S)$.*
2. *For every $(y_i)_1^n \in E_n^\nu(Y,S)$, there exists $(x_i)_1^n \in E_n^\mu(X,T)$ with $\pi(x_i) = y_i$, $i = 1,2,\ldots,n$.*

Proof: (1) follows directly from the definition. Using Theorem 4.9 and Theorem 4.4, (2) follows from the proof of Theorem 5 in [BGH]. ∎

5. Entropy of measurable covers and partitions

In this section, we prove that for a MDS $(Y,\mathcal{D},\nu,T)$, if $\mathcal{U}$ is a finite measurable cover of Y such that $h_\nu(T,\mathcal{U}) > 0$, then $h_c(T,\mathcal{U}) > 0$. In section 6 we will apply this result to show that each n-meas is an n-topo. First, we discuss some basic properties of the Pinsker factor.

We say $(Y,\mathcal{D},\nu,T)$ is a **Lebesgue system**, if $(Y,\mathcal{D},\nu)$ is a Lebesgue space and T is an invertible measure-preserving mapping on it. Here, we require that $\mathcal{D}$ is complete under ν, i.e., if $A \in \mathcal{D}$ with $\nu(A) = 0$ then for any $C \subset A$ one has $C \in \mathcal{D}$.

Let $(Y,\mathcal{D},\nu,T)$ be a Lebesgue system, P_ν be its Pinsker σ-algebra. Let $\pi\colon (Y,\mathcal{D},\nu,T) \to (Z,\mathcal{Z},\eta,T)$ be the measure-theoretical Pinsker factor of $(Y,\mathcal{D},\nu,T)$, where we require that $(Z,\mathcal{Z},\eta,T)$ is also a Lebesgue system.

Let $\nu = \int_Z \nu_z d\eta(z)$ be the disintegration of ν over (Z,η) (see [F] and [R]). It is known that for η-a.e. $z \in Z$, $\nu_z(\pi^{-1}(z)) = 1$.

Recall that (see section 4) for a MDS $(Y,\mathcal{D},\nu,T)$, $\lambda_n(\nu)$ is a measure on $(Y^{(n)},\mathcal{D}^{(n)},T^{(n)})$ with $\lambda_n(\nu)(\prod_{i=1}^n A_i) = \int_Y \prod_{i=1}^n \mathbb{E}(1_{A_i}|P_\nu)d\nu(y)$ for any $A_i \in \mathcal{D}, i = 1,2,\ldots,n$. Note that for a Lebesgue system $(Y,\mathcal{D},\nu,T)$, $\mathbb{E}(1_A|P_\nu)(y) = \nu_{\pi(y)}(A)$ for $A \in \mathcal{D}$ and ν-a.e. $y \in Y$. Moreover, we have

$$\lambda_n(\nu) = \underbrace{\nu \times_Z \nu \times_Z \cdots \times_Z \nu}_{n} = \int_Z \underbrace{\nu_z \times \nu_z \cdots \times \nu_z}_{n} d\eta(z).$$

Given $l \in \mathbb{N}$, let $S = T^l$, and $\mathbb{P}$ be a finite measurable partition of Y. Define a function $h_\nu(S, \mathbb{P}, z)$ η-a.e. on Z by the formula

$$h_\nu(S, \mathbb{P}, z) := \lim_{n \to +\infty} H_{\nu_z}(\mathbb{P} | S^{-1}\mathbb{P}^n),$$

where $\mathbb{P}^n = \bigvee_{i=0}^{n-1} S^{-i}\mathbb{P}$. It is not hard to see that $h_\nu(S, \mathbb{P}, z)$ is a measurable function on Z and $h_\nu(S, \mathbb{P}, z) \le \log \#(\mathbb{P})$. Moreover, η is invariant under S. We have

LEMMA 5.1: $\int_Z h_\nu(S, \mathbb{P}, z) d\eta(z) = h_\nu(S, \mathbb{P})$.

Proof: Using monotone convergence theorem and noting that $\lim_{n \to +\infty} a_n = a$ implies $\lim_{n \to +\infty} \frac{1}{n} \sum_{i=1}^n a_i = a$, we have

$$\begin{aligned} \int_Z h_\nu(S, \mathbb{P}, z) d\eta(z) &= \int_Z \lim_{n \to +\infty} H_{\nu_z}(\mathbb{P} | S^{-1}\mathbb{P}^n) d\eta(z) \\ &= \lim_{n \to +\infty} \int_Z H_{\nu_z}(\mathbb{P} | S^{-1}\mathbb{P}^n) d\eta(z) \\ &= \lim_{n \to +\infty} \int_Z H_{\nu_z}(\mathbb{P}^{n+1}) - H_{\nu_z}(S^{-1}\mathbb{P}^n) d\eta(z) \qquad (5.1) \\ &= \lim_{n \to +\infty} \int_Z H_{\nu_z}(\mathbb{P}^{n+1}) - H_{\nu_z}(\mathbb{P}^n) d\eta(z) \\ &= \lim_{n \to +\infty} \frac{1}{n} \int_Z H_{\nu_z}(\mathbb{P}^n) d\eta(z). \end{aligned}$$

For $A \in \mathbb{P}^n$, since $\mathbb{E}(1_A | P_\nu)(y) = \nu_{\pi(y)}(A)$ for ν-a.e. $y \in Y$, we have

$$\begin{aligned} H_\nu(\mathbb{P}^n | P_\nu) &= \sum_{A \in \mathbb{P}^n} \int_Y -\mathbb{E}(1_A | P_\nu)(y) \log(\mathbb{E}(1_A | P_\nu)(y)) d\nu(y) \\ &= \sum_{A \in \mathbb{P}^n} \int_Z -\nu_z(A) \log(\nu_z(A)) d\eta(z) \\ &= \int_Z H_{\nu_z}(\mathbb{P}^n) d\eta(z). \end{aligned}$$

Since P_ν is also the Pinsker σ-algebra of (Y, ν, S), (5.1) equals

$$\lim_{n \to +\infty} \frac{1}{n} H_\nu(\mathbb{P}^n | P_\nu) = h_\nu(S, \mathbb{P}).$$

This ends the proof of Lemma 5.1. ∎

The following lemma is from [R] (lemma 3′ in §4 No. 2).

LEMMA 5.2: *Suppose ν_z is non-atomic for η-a.e. $z \in Z$. If B is a measurable set of Y with $\nu_z(B) \geq r > 0$ for η-a.e. $z \in Z$, then for any $0 \leq \theta \leq r$ there exists a measurable set B_θ such that $B_\theta \subset B$ and $\nu_z(B_\theta) = \theta$ for η-a.e. $z \in Z$.*

With the help of Lemma 5.2 we prove a lemma which is important in the proof of Theorem 5.5 and could be useful in other settings.

LEMMA 5.3: *Suppose ν_z is non-atomic for η-a.e. $z \in Z$. If $U_i \in \mathcal{D}$, $i = 1, 2, \ldots, n$ with $\lambda_n(U_1 \times U_2 \times \cdots \times U_n) > 0$, then there exist a measurable set $A \subset Z$ with $\eta(A) > 0$, a positive integer $r > n$ and a measurable partition $\mathbb{P} = \{B_1, B_2, \ldots, B_r\}$ of Y such that $\pi^{-1}(A) \cap B_i \subset U_i$, $i = 1, 2, \ldots, n$ and $\nu_z(B_j) = 1/r$, $j = 1, 2, \ldots, r$ for η-a.e. $z \in Z$.*

Proof: Put $C_i = \{z \in Z : \nu_z(U_i) > 0\}$, $i = 1, 2, \ldots, n$. Since

$$0 < \lambda_n(\nu)(U_1 \times U_2 \times \cdots \times U_n) = \int_Z \prod_{i=1}^n \nu_z(U_i) d\eta(z),$$

one has $\eta(\bigcap_{i=1}^n C_i) > 0$.

We use induction to construct B_i and first we do so for B_1. Let $A' = \bigcap_{i=1}^n C_i$; then A' is a measurable set of Z and $\eta(A') > 0$. It is easy to see that there exist a positive integer $r > n$ and a measurable set $A \subset A'$ such that $\eta(A) > 0$ and $\nu_z(U_i) \geq n/r$ for any $z \in A$, $i = 1, 2, \ldots, n$. Setting

$$D_1 = \pi^{-1}(Z \setminus A) \cup (\pi^{-1}(A) \cap U_1),$$

we have $\nu_z(D_1) \geq n/r$ for η-a.e. $z \in Z$. By Lemma 5.2, there exists a measurable set $B_1 \subset D_1$ such that $\nu_z(B_1) = 1/r$ for η-a.e. $z \in Z$, and $B_1 \cap \pi^{-1}(A) \subset D_1 \cap \pi^{-1}(A) \subset U_1$.

Suppose measurable sets B_k have been constructed ($1 \leq k \leq r$), and B_k satisfies

$(1)_k$ for $1 \leq i \leq k-1, B_k \cap B_i = \emptyset$;

$(2)_k$ $\nu_z(B_k) = 1/r$ for η-a.e. $z \in Z$;

$(3)_k$ $B_k \cap \pi^{-1}(A) \subset U_k$ (when $k > n$, we set $U_k = Y$).

If $k = r$, we are done. If $k < r$, we set

$$D_{k+1} = \begin{cases} (\pi^{-1}(Z \setminus A) \cup (\pi^{-1}(A) \cap U_{k+1})) \setminus \bigcup_{i=1}^k B_i & \text{if } k+1 \leq n, \\ Y \setminus \bigcup_{i=1}^k B_i & \text{if } k+1 > n. \end{cases}$$

It is not hard to see that $\nu_z(D_{k+1}) \geq 1/r$ for η-a.e. $z \in Z$. By Lemma 5.2, we can find measurable sets $B_{k+1} \subset D_{k+1}$ such that $\nu_z(B_{k+1}) = 1/r$ for η-a.e. $z \in Z$. Obviously, B_{k+1} satisfies $(1)_{k+1}, (2)_{k+1}, (3)_{k+1}$.

By the above inductive construction, one gets a partition $\mathbb{P}=\{B_1,B_2,\dots,B_r\}$ of Y, and $\mathbb{P}$ satisfies $\nu_z(B_k) = 1/r$, $k = 1,2,\dots,r$ for η-a.e. $z \in Z$ and $B_i \cap \pi^{-1}(A) \subset U_i$, $i = 1,\dots,n$. ∎

The following lemma is well known.

LEMMA 5.4: *Let $(Y,\mathcal{D},\nu,T)$ be a Lebesgue system, $\pi\colon (Y,\mathcal{D},\nu,T)\to(Z,\mathcal{Z},\eta,T)$ be the Pinsker factor of $(Y,\mathcal{D},\nu,T)$ and $\int_Z \nu_z d\eta = \nu$ be the disintegration of ν over (Z,η). If ν is an ergodic measure with $h_\nu(T) > 0$, then we have*

1. *ν_z is non-atomic for η-a.e. $z \in Z$;*
2. *$\pi\colon (Y,\mathcal{D},\nu,T) \to (Z,\mathcal{Z},\eta,T)$ is a weakly mixing extension and $\lambda_n(\nu)$ is ergodic.*

With the above preparation we can show Theorem 5.5. In fact, Theorem 5.5 is a consequence of Theorem 5.7. The proof presented here has some important by-products which will be used in later sections.

THEOREM 5.5: *Let (X,T) be a TDS and $\mu \in M(X,T)$. If $\mathcal{U}$ is a Borel cover of X such that $h_\mu(T,\mathcal{U}) > 0$, then $h_c(T,\mathcal{U}) > 0$.*

Proof: By Lemma 4.8, we may assume that μ is ergodic. Let $\mathcal{B}_\mu$ be the completion of the Borel σ-algebra $\mathcal{B}(X)$ under μ. It is well known that $(X,\mathcal{B}_\mu,\mu,T)$ is a Lebesgue system. Let P_μ be the Pinsker σ-algebra of $(X,\mathcal{B}_\mu,\mu,T)$. Let $\pi\colon (X,\mathcal{B}_\mu,\mu,T) \to (Z,\mathcal{Z},\eta,T)$ be the Pinsker factor of $(X,\mathcal{B}_\mu,\mu,T)$ and $\mu = \int_Z \mu_z d\eta$ be the disintegration of μ over (Z,η). By Lemma 5.4, μ_z is non-atomic for η-a.e. $z \in Z$.

Let $\mathcal{U} = \{U_1^c,U_2^c,\dots,U_n^c\}$. By Corollary 4.7, $\lambda_n(\mu)(\prod_{i=1}^n U_i) > 0$. As μ_z is non-atomic for η-a.e. $z \in Z$ and $\lambda_n(\mu)(\prod_{i=1}^n U_i) > 0$, by Lemma 5.3 there exist a measurable set $A \subset Z$ with $\eta(A) > 0$, a positive integer $r > n$ and a measurable partition $\mathbb{P} = \{B_1,B_2,\dots,B_r\}$ of X such that $\pi^{-1}(A)\cap B_i \subset U_i$, $i = 1,2,\dots,n$ and $\mu_z(B_j) = 1/r$, $j = 1,2,\dots,r$ for η-a.e. $z \in Z$.

Since

$$\lim_{m\to+\infty} h_\mu(T^m,\mathbb{P}) = H_\mu(\mathbb{P}|P_\mu) = \sum_{j=1}^{r}\int_Z -\mu_z(B_j)\log\mu_z(B_j)d\eta = \log r,$$

we can find $l > 0$ such that $h_\mu(T^l,\mathbb{P}) > \eta(Z\setminus A)\cdot\log r + \eta(A)\cdot\log(r-1)$.

Let $S = T^l$ and define η-a.e. on Z a function $h_\mu(S,\mathbb{P},z)$. By Lemma 5.1, we have $\int_Z h_\mu(S,\mathbb{P},z)d\eta(z) = h_\mu(S,\mathbb{P})$.

For $z \in Z$ set

$$f(z)\colon = (h_\mu(S,\mathbb{P},z) - \log(r-1))\cdot 1_A(z).$$

As $h_\mu(S,\mathbb{P},z)\le \log r$, $f(z)$ is a bounded measurable function on Z and

$$1_A(z)\ge \frac{f(z)}{\log(\frac{r}{r-1})}.$$

Note that

$$\begin{aligned}\int_Z f(z)d\eta(z) &= \int_A h_\mu(S,\mathbb{P},z)d\eta(z) - \eta(A)\cdot\log(r-1)\\ &= \int_Z h_\mu(S,\mathbb{P},z)d\eta(z) - \int_{Z\setminus A} h_\mu(S,\mathbb{P},z)d\eta(z) - \eta(A)\cdot\log(r-1)\\ &\ge h_\mu(S,\mathbb{P}) - \eta(Z\setminus A)\cdot\log r - \eta(A)\cdot\log(r-1) > 0.\end{aligned}$$

By the Birkhoff ergodic theorem, $\frac{1}{m}\sum_{i=0}^{m-1} f(S^i z)$ converges a.e. to a function $f^\star\in L^1(\eta)$ and $\int_Z f^\star(z)d\eta(z)=\int_Z f(z)d\eta(z)>0$.

Set $R=\{z\in Z: f^\star(z)>0\}$. Then one gets $\eta(R)>0$. It is well known that for any $i\in\mathbb{N}$, $S^i\mu_z=\mu_{S^iz}$ for η-a.e. $z\in Z$. By the Birkhoff ergodic theorem, $\frac{1}{m}\sum_{i=0}^{m-1} 1_A(S^iz)$ converges a.e. to a function $1_A{}^\star\in L^1(\eta)$.

Choose $\omega\in R$ with $S^i\mu_\omega=\mu_{S^i\omega}$, $S^i\mathbb{P}\subset\mathcal{B}_{\mu_\omega}$ for any $i\in\mathbb{Z}$ and

$$\lim_{m\to+\infty}\frac{1}{m+1}\sum_{i=0}^{m} 1_A(S^i\omega)=1_A{}^\star(\omega)$$

and let

$$\mathcal{A}=\{k\in\mathbb{Z}_+ : S^k\omega\in A\}:=\{a_1<a_2<\cdots\},$$

where $\mathcal{B}_{\mu_\omega}$ is the completion of $\mathcal{B}$ under μ_ω. Put

$$\mathcal{V}=\{(\pi^{-1}(A)\cap B_1)^c,(\pi^{-1}(A)\cap B_2)^c,\ldots,(\pi^{-1}(A)\cap B_r)^c\}.$$

Then $\mathcal{U}\ge\mathcal{V}$ and we have the following Claim whose proof comes a little bit later.

CLAIM: $h_c(S,\mathcal{V})\ge f^\star(\omega)>0$.

With this Claim finally, we have

$$h_c(T,\mathcal{U})\ge\frac{1}{l}h_c(S,\mathcal{U})\ge\frac{1}{l}h_c(S,\mathcal{V})\ge\frac{1}{l}f^\star(\omega)>0. \qquad\blacksquare$$

Proof of Claim: For any $G\subset X$, and any cover $\mathcal{E}=\{E_1,E_2,\ldots,E_t\}$ of X, let $G\cap\mathcal{E}=\{G\cap E_1,G\cap E_2,\ldots,G\cap E_t\}$.

Note that for any $x \in \pi^{-1}(\omega)$, $S^{a_i}x \in \pi^{-1}(A)$. Therefore, for any $x \in \pi^{-1}(\omega), i \in \mathbb{N}$, and any subset D of X, we have $S^{a_i}(x) \in D$ if and only if $S^{a_i}(x) \in \pi^{-1}(A) \cap D$. Moreover, for any finite cover $\mathcal{D}$ of X and $k \in \mathbb{N}$, we have

$$N(\bigvee_{i=1}^{k} S^{-a_i}\mathcal{D}, \pi^{-1}(\omega)) = N(\bigvee_{i=1}^{k} S^{-a_i}(\pi^{-1}(A) \cap \mathcal{D}), \pi^{-1}(\omega)).$$

Let $\mathcal{V}' = \{\pi^{-1}(A) \cap B_1, \pi^{-1}(A) \cap B_2, \ldots, \pi^{-1}(A) \cap B_r\}$. Since $\pi^{-1}(A) \cap \mathcal{V} = (r-1)\mathcal{V}'$ and $\pi^{-1}(A) \cap \mathbb{P} = \mathcal{V}'$, we have

$$\begin{aligned} H((\mathcal{V})_0^{m-1}(S)) &= \log N((\mathcal{V})_0^{m-1}(S)) \geq \log N((\mathcal{V})_0^{m-1}(S), \pi^{-1}(\omega)) \\ &\geq \log N(\vee_{i=1}^{k_m} S^{-a_i}\mathcal{V}, \pi^{-1}(\omega)) \\ &\qquad (\text{where } a_{k_m} \leq m-1, a_{k_m+1} \geq m) \\ &= \log N(\vee_{i=1}^{k_m} S^{-a_i}(\pi^{-1}(A) \cap \mathcal{V}), \pi^{-1}(\omega)) \\ &= \log N(\vee_{i=1}^{k_m} S^{-a_i}((r-1)\mathcal{V}'), \pi^{-1}(\omega)) \\ &\geq \log N(\vee_{i=1}^{k_m} S^{-a_i}\mathcal{V}', \pi^{-1}(\omega)) - k_m \cdot \log(r-1) \\ &\qquad\quad (\text{by Lemma 3.3}) \\ &= \log N(\vee_{i=1}^{k_m} S^{-a_i}(\pi^{-1}(A) \cap \mathbb{P}), \pi^{-1}(\omega)) - k_m \cdot \log(r-1) \\ &= \log N(\vee_{i=1}^{k_m} S^{-a_i}\mathbb{P}, \pi^{-1}(\omega)) - k_m \cdot \log(r-1). \end{aligned}$$

Now, we estimate $M(l) = \log N(S^{-a_1}\mathbb{P} \vee \cdots \vee S^{-a_l}\mathbb{P}, \pi^{-1}(\omega))$. Since for any $i \in \mathbb{N}$, $S^i\mu_\omega = \mu_{S^i\omega}$, we have $H_{\mu_\omega}(S^{-k}\mathbb{P}|\mathcal{C}) = H_{\mu_{S^k\omega}}(\mathbb{P}|S^k\mathcal{C})$, where $\mathcal{C}, S^k\mathcal{C}$ are finite subalgebras of $\mathcal{B}_{\mu_\omega}$ and $k \in \mathbb{Z}_+$. Thus,

$$\begin{aligned} M(m) &= \log \#\{(j_1, \ldots, j_l) \in \{1, \ldots, r\}^l \colon \pi^{-1}(\omega) \cap (\cap_{i=1}^{l} S^{-a_i}B_{j_i}) \neq \emptyset\} \\ &\geq \log \#\{(j_1, \ldots, j_l) \in \{1, \ldots, r\}^l \colon \mu_\omega(\pi^{-1}(\omega) \cap (\cap_{i=1}^{l} S^{-a_i}B_{j_i})) > 0\} \\ &\geq H_{\mu_\omega}(\bigvee_{i=1}^{l} S^{-a_i}\mathbb{P}) \\ &= H_{\mu_\omega}(S^{-a_1}\mathbb{P}|\bigvee_{i=2}^{l} S^{-a_i}\mathbb{P}) + H_{\mu_\omega}(\bigvee_{i=2}^{l} S^{-a_i}\mathbb{P}) \\ &= H_{\mu_{S^{a_1}\omega}}(\mathbb{P}|\bigvee_{i=1}^{l} S^{-(a_i-a_1)}\mathbb{P}) + H_{\mu_\omega}(\bigvee_{i=2}^{l} S^{-a_i}\mathbb{P}) \\ &\geq h_\mu(S, \mathbb{P}, S^{a_1}\omega) + H_{\mu_\omega}(\bigvee_{i=2}^{l} S^{-a_i}\mathbb{P}) \geq \sum_{i=1}^{l} h_\mu(S, \mathbb{P}, S^{a_i}\omega). \end{aligned}$$

Combining the above results, we have

$$\begin{aligned} H((\mathcal{V})_0^{m-1}(S)) &\geq \sum_{i=1}^{k_m} h_\mu(S,\mathbb{P},S^{a_i}\omega) - k_m \cdot \log(r-1) \\ &= \sum_{i=0}^{m-1} (h_\mu(S,\mathbb{P},S^i\omega) - \log(r-1)) \cdot 1_A(S^i\omega) \\ &= \sum_{i=0}^{m-1} f(S^i\omega). \end{aligned}$$

Therefore, one gets

$$\begin{aligned} h_c(S,\mathcal{V}) &= \lim_{m\to+\infty} \frac{1}{m} H((\mathcal{V})_0^{m-1}(S)) \\ &\geq \lim_{m\to+\infty} \frac{1}{m} \Sigma_{i=0}^{m-1} f(S^i\omega) = f^\star(\omega). \end{aligned}$$

This ends the proof of Claim. ∎

Remark: Since

$$\begin{aligned} 1_A{}^\star(\omega) &= \lim_{m\to+\infty} \frac{1}{m+1} \sum_{i=0}^{m} 1_A(S^i\omega) \\ &\geq \lim_{m\to+\infty} \frac{1}{m+1} \sum_{i=0}^{m} \frac{f(S^i\omega)}{\log(\frac{r}{r-1})} = \frac{f^\star(\omega)}{\log(\frac{r}{r-1})} > 0 \end{aligned}$$

and

$$\begin{aligned} \lim_{m\to+\infty} \frac{|\mathcal{A}\cap\{0,1,2,\ldots,m\}|}{m+1} &= \lim_{m\to+\infty} \frac{1}{m+1} \sum_{i=0}^{m} 1_A(S^i\omega) \\ &= 1_A{}^\star(\omega), \end{aligned}$$

$\mathcal{A}$ is a subset of $\mathbb{Z}_+$ with positive density.

Let $Q(k) = S^{-a_1}(\pi^{-1}(A)\cap\mathbb{P}) \vee \cdots \vee S^{-a_k}(\pi^{-1}(A)\cap\mathbb{P})$ and $|Q(k)|$ its cardinality. From the proof of Claim, one has

$$\begin{aligned} \log|Q(k)| &\geq \log N(Q(k),\pi^{-1}(\omega)) = M(k) \\ &\geq \sum_{i=1}^{k} h_\mu(S,\mathbb{P},S^{a_i}\omega) \\ &= k\cdot\log(r-1) + \sum_{j=0}^{a_k} (h_\mu(S,\mathbb{P},S^j\omega) - \log(r-1)) \cdot 1_A(S^j\omega) \\ &= k\cdot\log(r-1) + \sum_{j=0}^{a_k} f(S^j\omega). \end{aligned}$$

Therefore,

$$|Q(k)| \geq (r-1)^k 2^{\sum_{j=0}^{a_k} f(S^j\omega)} = (r-1)^k 2^{k \frac{a_k}{k} \frac{1}{a_k} \sum_{j=0}^{a_k} f(S^j\omega)}.$$

Since

$$\lim_{k\to+\infty} \frac{1}{a_k} \sum_{j=0}^{a_k} f(S^j\omega) = f^\star(\omega) > 0$$

and

$$\lim_{k\to+\infty} (k/a_k) = \lim_{m\to+\infty} \frac{1}{m} \sum_{j=0}^{m-1} 1_A(S^j\omega) > 0,$$

there exist $h > 0$ and $M \in \mathbb{N}$ such that for any $k \geq M$,

$$|S^{-a_1}(\pi^{-1}(A) \cap \mathbb{P}) \vee \cdots \vee S^{-a_k}(\pi^{-1}(A) \cap \mathbb{P})| \geq (r-1)^k 2^{kh}. \quad ∎$$

Set $D = \pi^{-1}(A)$ and $\mathcal{S} = \{s_i\colon s_i = la_i\}$. Then S is a subset of $\mathbb{Z}_+$ with positive density. Moreover, one has the following corollary, which will be used to characterize the topological entropy n-tuples.

COROLLARY 5.6: *Let (X,T) be a TDS and $\mu \in M(X,T)$ be an ergodic measure with $h_\mu(T) > 0$. If $U_i \in \mathcal{B}_\mu$, $i = 1,2,\ldots,n$ with $\lambda_n(\mu)(\prod_{i=1}^n U_i) > 0$, then there exist a measurable partition $\mathbb{P} = \{B_1, B_2, \ldots, B_r\}$ of X with $r > n$, a measurable set $D \subset X$ and a positive density subset $S = \{s_1, s_2, \ldots\}$ of $\mathbb{Z}_+$ such that*

(1) *$D \cap B_i \subset U_i$, $i = 1,2,\ldots,n$,*

(2) *there exist $h > 0$ and $M \in \mathbb{N}$ such that for any $k \geq M$,*

$$|T^{-s_1}(D \cap \mathbb{P}) \vee T^{-s_2}(D \cap \mathbb{P}) \vee \cdots \vee T^{-s_k}(D \cap \mathbb{P})| \geq (r-1)^k 2^{kh}.$$

Proof: The corollary now follows from the proof Theorem 5.5 and the above Remark. ∎

We may use the claim in the proof of Theorem 4.6 to prove a result stronger than Theorem 5.5.

THEOREM 5.7: *Let $(Y, \mathcal{D}, \nu, T)$ be a MDS. If $\mathcal{U} = \{U_1, U_2, \ldots, U_n\}$ $(n \geq 2)$ is a measurable cover of Y with $h_\nu(T,\mathcal{U}) > 0$, then $h_c(T,\mathcal{U}) > 0$.*

Proof: Let P_ν be the Pinsker σ-algebra of $(Y, \mathcal{D}, \nu, T)$. For any

$$s = (s(1), \ldots, s(n)) \in \{0,1\}^n,$$

set $A_s = \bigcap_{i=1}^n U_i(s(i))$, where $U_i(0) = U_i$ and $U_i(1) = Y \setminus U_i$, and $\alpha = \{A_s\colon s \in \{0,1\}^n\}$.

By the claim in the proof of Theorem 4.6, there exists $\epsilon > 0$ such that $H_\nu(\alpha|\beta \vee P_\nu) \leq H_\nu(\alpha|P_\nu) - \epsilon$, for any finite measurable partition β which is finer than $\mathcal{U}$ as a cover.

Since $\lim_{n\to+\infty} h_\nu(T^n,\alpha) = H_\nu(\alpha|P_\nu)$, there exists $l > 0$ such that $h_\nu(T^l,\alpha) > H_\nu(\alpha|P_\nu) - \epsilon/2$. Set $S = T^l$.

Let $n \in \mathbb{N}$ and β be a finite measurable partition with $\beta \succeq \bigvee_{i=0}^{n-1} S^{-i}\mathcal{U}$. Since $S^i\beta$ is finer than $\mathcal{U}$ for $i \in \{0,1,\ldots,n-1\}$, one has

$$\begin{aligned} H_\nu(\beta|P_\nu) &= H_\nu\Big(\beta \vee \bigvee_{i=0}^{n-1} S^{-i}\alpha|P_\nu\Big) - H_\nu\Big(\bigvee_{i=0}^{n-1} S^{-i}\alpha|\beta \vee P_\nu\Big) \\ &\geq H_\nu\Big(\bigvee_{i=0}^{n-1} S^{-i}\alpha|P_\nu\Big) - \sum_{i=0}^{n-1} H_\nu(\alpha|S^i\beta \vee P_\nu) \\ &\geq H_\nu\Big(\bigvee_{i=0}^{n-1} S^{-i}\alpha|P_\nu\Big) - n(H_\nu(\alpha|P_\nu) - \epsilon). \end{aligned}$$

For each $n \in \mathbb{N}$, there exists a finite measurable partition β_n, finer than $\bigvee_{i=0}^{n-1} S^{-i}\mathcal{U}$, such that

$$H_\nu(\beta_n|P_\nu) \leq H_\nu(\beta_n) \leq \log N\Big(\bigvee_{i=0}^{n-1} S^{-i}\mathcal{U}\Big).$$

Hence

$$\begin{aligned} h_c(S,\mathcal{U}) &= \lim_{n\to\infty} \frac{1}{n} \log N\Big(\bigvee_{i=0}^{n-1} S^{-i}\mathcal{U}\Big) \geq \limsup_{n\to\infty} \frac{1}{n} H_\nu(\beta_n|P_\nu) \\ &\geq \limsup_{n\to\infty} \frac{1}{n}\Big[H_\nu\Big(\bigvee_{i=0}^{n-1} S^{-i}\alpha|P_\nu\Big) - n(H_\nu(\alpha|P_\nu) - \epsilon)\Big] \\ &= h_\nu(S,\alpha) - H_\nu(\alpha|P_\nu) + \epsilon > \epsilon/2. \end{aligned}$$

Thus, $h_c(T,\mathcal{U}) \geq \frac{1}{l} h_c(S,\mathcal{U}) > 0$ and the proof is finished. ∎

6. The relations of topological and measure entropy n-tuples

Let (X,T) be a TDS. An n-topo $(x_i)_1^n$ is called **intrinsic** if $x_i \neq x_j$ for $i \neq j$. For $n \geq 2$, denote by $E_n^e(X,T)$ the set of intrinsic n-topo. It is easy to see that $E_n(X,T)$, $E_n'(X,T)$, and $E_n^e(X,T)$ are invariant under the change of coordinates in $X^{(n)}$.

In this section, we shall prove that if $\mu \in M(X,T)$, then $E_n(X,T) \supseteq E_n^\mu(X,T)$ for each $n \geq 2$ and there exists $\mu \in M(X,T)$ such that $E_n(X,T) = E_n^\mu(X,T)$

for each $n \geq 2$. Moreover, we shall show that a TDS with positive topological entropy must have intrinsic n-topo and $E_n(X,T) = \mathrm{cl}(E_n^e(X,T)) \setminus \Delta_n(X)$ for each $n \geq 2$. First we prove

THEOREM 6.1: *Let (X,T) be a TDS and $\mu \in M(X,T)$. Then for each $n \geq 2$,*

$$E_n(X,T) \supseteq E_n^\mu(X,T) = \mathrm{Supp}(\lambda_n(\mu)) \setminus \Delta_n(X).$$

Proof: Let $(x_i)_{i=1}^n \in E_n^\mu(X,T)$ and $\mathcal{U}$ be a finite open cover of X admissible with respect to $(x_i)_{i=1}^n$. It is easy to see that any finite measurable partition $\mathbb{P}$, finer than $\mathcal{U}$, is an admissible partition with respect to $(x_i)_{i=1}^n$. Thus $h_\mu(T,\mathbb{P}) > 0$, since $(x_i)_1^n \in E_n^\mu(X,T)$. By Theorem 5.5, $h_{\mathrm{top}}(T,\mathcal{U}) = h_c(T,\mathcal{U}) > 0$. ∎

LEMMA 6.2: *Let (X,T) be a TDS. If $\mu \in M(X,T)$ is ergodic, then for every $n \geq 2$,*

$$\mathrm{cl}(E_n^e(X,T)) \setminus \Delta_n(X) \supset \mathrm{Supp}(\lambda_n(\mu)) \setminus \Delta_n(X) E_n^\mu(X,T).$$

In particular, if $h_{\mathrm{top}}(T) > 0$, then $E_n^e(X,T) \neq \emptyset$ for every $n \geq 2$.

Proof: Let $\mathcal{B}_\mu$ be the completion of $\mathcal{B}(X)$ under μ. Then $(X,\mathcal{B}_\mu,\mu,T)$ is a Lebesgue system.

If $h_\mu(T) = 0$, one has $\mathrm{Supp}(\lambda_n(\mu)) \setminus \Delta_n(X) = E_n^\mu(X,T) = \emptyset$.

We now assume $h_\mu(T) > 0$. Let P_μ be the Pinsker σ-algebra of $(X,\mathcal{B}_\mu,\mu,T)$. Let $\pi\colon (X,\mathcal{B}_\mu,\mu,T) \to (Z,\mathcal{Z},\eta,T)$ be the Pinsker factor of $(X,\mathcal{B}_\mu,\mu,T)$ and $\mu = \int_Z \mu_z d\eta$ be the disintegration of μ over (Z,η). By Lemma 5.4, μ_z is non-atomic for η-a.e. $z \in Z$ and hence

$$\lambda_n(\mu)(\Delta_n) = \int_Z \mu_z \times \mu_z \times \cdots \times \mu_z(\Delta_n) d\eta(z) = 0.$$

Thus, $\mathrm{Supp}(\lambda_n(\mu)) \setminus \Delta_n \neq \emptyset$. Since μ is ergodic, π is a weakly mixing extension and $\lambda_n(\mu)$ is ergodic. Thus $(\mathrm{Supp}(\lambda_n(\mu)), T^{(n)})$ is topologically transitive. Since $\mathrm{Supp}(\lambda_n(\mu))$ is invariant under exchanging coordinates of $X^{(n)}$, for each transitive point $(x_i)_1^n$ of $(\mathrm{Supp}(\lambda_n(\mu)), T^{(n)})$ we have $x_i \neq x_j$ if $i \neq j$.

By Theorem 6.1, $(x_i)_1^n \in E_n^e(X,T)$. Since $\mathrm{cl}(E_n^e(X,T))$ is a closed $T^{(n)}$-invariant subset of $X^{(n)}$, one has $\mathrm{cl}(E_n^e(X,T)) \setminus \Delta_n(X) \supset \mathrm{Supp}(\lambda_n(\mu)) \setminus \Delta_n(X)$ $(= E_n^\mu(X,T))$. ∎

We say that a partition $\mathbb{P}$ is finer than a cover $\mathcal{U}$ when every atom of $\mathbb{P}$ is contained in an element of $\mathcal{U}$. The following lemma is Theorem 1 of [BGH].

LEMMA 6.3: *Let (X,T) be a TDS, and $\mathcal{U}$ an open cover of X. Then there exists $\mu \in M(X,T)$ such that $h_\mu(T,\mathbb{P}) \geq h_{\text{top}}(T,\mathcal{U})$ for all Borel partitions $\mathbb{P}$ finer than $\mathcal{U}$, i.e., $h_\mu(T,\mathcal{U}) \geq h_{\text{top}}(T,\mathcal{U})$.*

With the above preparation we now show

THEOREM 6.4: *Let (X,T) be a TDS. Then there exists $\mu \in M(X,T)$ such that $E_n(X,T) = E_n^\mu(X,T)(= \text{Supp}(\lambda_n(\mu)) \setminus \Delta_n(X))$ for each $n \geq 2$.*

Proof: Let $n \geq 2$. First we have

CLAIM: *For any point $(x_i)_1^n \in E_n(X,T)$ and any neighborhood U_i of x_i, there exists $\nu \in M(X,T)$ with $E_n^\nu(X,T) \cap (U_1 \times U_2 \times \cdots \times U_n) \neq \emptyset$.*

Proof of Claim: Without loss of generality, assume that U_i is a closed neighborhood of x_i such that $U_i \cap U_j = \emptyset$ if $x_i \neq x_j$ and $U_i = U_j$ if $x_i = x_j$, $1 \leq i < j \leq n$. Let $\mathcal{U} = \{U_1^c, U_2^c, \ldots, U_n^c\}$. Clearly, $h_{\text{top}}(T,\mathcal{U}) > 0$. By Lemma 6.3, there exists $\nu \in M(X,T)$ such that $h_\nu(T,\mathbb{P}) \geq h_{\text{top}}(T,\mathcal{U})$ for all Borel partitions $\mathbb{P}$ finer than $\mathcal{U}$. By Lemma 4.3 one has $\lambda_n(\nu)(\prod_{i=1}^n U_i) > 0$, i.e., $\text{Supp}(\lambda_n(\nu)) \cap \prod_{i=1}^n U_i \neq \emptyset$. As $\prod_{i=1}^n U_i \cap \Delta_n(X) = \emptyset$, one has $E_n^\nu(X,T) \cap (U_1 \times U_2 \times \cdots \times U_n) \neq \emptyset$ by Theorem 4.4. This ends the proof of the claim.

By the claim, we can choose a dense sequence of points

$$\{(x_1^m, x_2^m, \ldots, x_n^m) \colon m \geq 1\}$$

in $E_n(X,T)$ with $(x_1^m, x_2^m, \ldots, x_n^m) \in E_n^{\nu_n^m}(X,T)$ for some $\nu_n^m \in \mathcal{M}(X,T)$.

Let

$$\mu = \sum_{n=2}^{+\infty} \frac{1}{2^{n-1}} \left(\sum_{m=1}^{+\infty} \frac{1}{2^m} \nu_n^m \right).$$

Since for any measurable partition α of X, $n \geq 2$ and $m \in \mathbb{N}$,

$$h_\mu(T,\alpha) \geq \frac{1}{2^{m+n-1}} h_{\nu_n^m}(T,\alpha),$$

therefore $E_n^{\nu_n^m}(X,T) \subset E_n^\mu(X,T)$. In particular, $(x_1^m, x_2^m, \ldots, x_n^m) \in E_n^\mu(X,T)$. Moreover,

$$E_n^\mu(X,T) \supset \overline{\{(x_1^m, x_2^m, \ldots, x_n^m); m \geq 1\}} \setminus \Delta_n(X) = E_n(X,T),$$

that is, $E_n^\mu(X,T) = E_n(X,T)$. ∎

THEOREM 6.5: *Let (X,T) be a TDS. Then for each $n \geq 2$,*

$$E_n(X,T) = \mathrm{cl}(E_n^e(X,T)) \setminus \Delta_n(X).$$

Proof: By Theorem 6.4, there exists $\mu \in M(X,T)$ such that $E_n(X,T) = E_n^\mu(X,T)$ for each $n \geq 2$. Let $\mu = \int_\Omega \mu_\omega dm(\omega)$ be the ergodic decomposition. By Theorem 4.9, for an appropriate choice of Ω,

$$\mathrm{cl}(\bigcup\{E_n^{\mu_\omega}(X,T)\colon \omega \in \Omega\}) \setminus \Delta_n = E_n^\mu(X,T).$$

Now for each $\omega \in \Omega$, $E_n^{\mu_\omega}(X,T) \subset \mathrm{cl}(E_n^e(X,T))$ by Lemma 6.2. Therefore, $E_n(X,T) \subset \mathrm{cl}(E_n^e(X,T))$. As $\mathrm{cl}(E_n^e(X,T)) \subset E_n'(X,T) \subset E_n(X,T) \cup \Delta_n(X)$, one has $E_n(X,T) = \mathrm{cl}(E_n^e(X,T)) \setminus \Delta_n(X)$. ∎

To end the section we give a characterization of n-topo. Let (X,T) be a TDS and $\mathbb{P} = \{A_1, A_2, \ldots, A_n\}$ be a measurable partition of X. We say that $\mathbb{P}$ is **topological non-trivial**, if $\mathrm{cl}(A_i) \neq X$ for each $1 \leq i \leq n$.

LEMMA 6.6: *Let (X,T) be a TDS and $\mu \in M(X,T)$. Then $E_n^\mu(X,T) = X^{(n)} \setminus \Delta_n(X)$ if and only if for any topological non-trivial measurable partition $\mathbb{P}$ of X by n sets, one has $h_\mu(T,\mathbb{P}) > 0$, where $n \geq 2$.*

Proof: Let $E_n^\mu(X,T) = X^{(n)} \setminus \Delta_n(X)$. For any topological non-trivial measurable partition $\mathbb{P} = \{A_1, A_2, \ldots, A_n\}$ of X, choose $x_i \in X \setminus \mathrm{cl}(A_i)$, $i = 1, 2, \ldots, n$. Clearly, $(x_i)_1^n \in X^{(n)} \setminus \Delta_n(X)$ and $\mathbb{P}$ is an admissible partition with respect to $(x_i)_1^n$. Thus $h_\mu(T,\mathbb{P}) > 0$.

Conversely, assume for any topological non-trivial measurable partition $\mathbb{P}$ of X by n sets that one has $h_\mu(T,\mathbb{P}) > 0$. Let $(x_i)_1^n \in X^{(n)} \setminus \Delta_n(X)$. For any admissible partition $\mathbb{P} = \{A_1, A_2, \ldots, A_n\}$ with respect to $(x_i)_1^n$, $\mathbb{P}$ is topological non-trivial. Thus $h_\mu(T,\mathbb{P}) > 0$. This shows $(x_i)_1^n \in E_n^\mu(X,T)$. ∎

THEOREM 6.7: *Let (X,T) be a TDS and $n \geq 2$. Then*

(1) *(X,T) has u.p.e. of order n if and only if there is $\mu \in M(X,T)$ such that for any topological non-trivial measurable partition $\mathbb{P}$ of X by n sets, one has $h_\mu(T,\mathbb{P}) > 0$.*
(2) *(X,T) has u.p.e. of all orders if and only if there is $\mu \in M(X,T)$ such that for any topological non-trivial measurable partition $\mathbb{P}$ of X by finite sets, one has $h_\mu(T,\mathbb{P}) > 0$.*

Proof: It is a direct consequence of Theorem 6.4 and Lemma 6.6. ∎

7. A theorem on weak disjointness

In the section, we shall show that u.p.e. of order 2 systems is weakly disjoint from any transitive system, and while doing this, we give another characterization for u.p.e. of order n or u.p.e. of all order systems. First, we need a combinatorial lemma, the idea of whose proof is in Proposition 8.2 [W1]; see also [S].

LEMMA 7.1: *Let $r \geq 2$. For every $h > 0$, there exist $b(h) > 0$ and $M_h \in \mathbb{N}$ such that if $m \geq M_h$ and $B \subset \{1, 2, \ldots, r\}^m$ satisfies $|B| \geq (r-1)^m 2^{mh}$, then we can find $J_m \subset \{1, 2, \ldots, m\}$ with $|J_m| \geq b(h)m$ and $B|_{J_m} = \{1, 2, \ldots, r\}^{J_m}$, i.e., for any $s \in \{1, 2, \ldots, r\}^{J_m}$ there is $b \in B$ with $b(j) = s(j)$ for $j \in J_m$.*

Definition 7.2: Let (X, T) be a topological dynamical system.

(1) Nonempty subsets $U_1, U_2, \ldots, U_n$ of X have Property P_n, if there exist $b > 0$ and $M_b \in \mathbb{N}$ such that for any natural number $m \geq M_b$, we can find $J_m \subset \{1, 2, \ldots, m\}$ with $|J_m| \geq bm$, and J_m satisfies that for any $s \in \{1, 2, \ldots, n\}^{J_m}$, there exists $x \in X$ with $x \in \bigcap_{j \in J_m} T^{-j} U_{s(j)}$.
(2) (X, T) is said to have Property P_n if any nonempty open subsets $U_1, U_2, \ldots, U_n$ of X have Property P_n. (X, T) is said to have strong property P if it has property P_n for every $n \geq 2$.

THEOREM 7.3: *Let (X, T) be a TDS and $(x_i)_1^n \notin \Delta_n(X)$ with $n \geq 2$. Then $(x_i)_1^n \in E_n(X, T)$ if and only if for any neighborhood $U_1 \times U_2 \times \cdots \times U_n$ of $(x_i)_1^n$, $U_1, U_2, \ldots, U_n$ have Property P_n.*

Proof: For any $(y_i)_1^m \in X^{(m)} \setminus \Delta_m$ with $m \geq 2$, assume that $\{y_1, y_2, \ldots, y_m\} = \{x_1, x_2, \ldots, x_n\}$ and $x_i \neq x_j$ for $1 \leq i < j \leq n$. It is clear $n \geq 2$. Note that $(y_i)_1^i \in E_m(X, T)$ if and only if $(x_i)_1^n \in E_n(X, T)$, and for any neighborhood V_i of y_i, $V_1, V_2, \ldots, V_m$ have Property P_m if and only if for any neighborhood U_i of x_i, $U_1, U_2, \ldots, U_n$ have Property P_n.

Thus, we may assume that $x_i \neq x_j$ for $1 \leq i < j \leq n$. Let $(x_i)_1^n \in E_n(X, T)$ and U_i be a neighborhood of x_i with $U_i \cap U_j = \emptyset$, $i \neq j$. By Theorem 6.4, there exists $\nu \in M(X, T)$ with $(x_i)_1^n \in \mathrm{Supp}(\lambda_n(\nu))$. Hence $\lambda_n(\nu)(\Pi_{i=1}^n U_i) > 0$. By Corollary 4.7, $h_\nu(T, \{U_1^c, U_2^c, \ldots, U_n^c\}) > 0$.

By Lemma 4.8, there exists an ergodic measure μ with $h_\mu(T, \{U_1^c, U_2^c, \ldots, U_n^c\}) > 0$, i.e., $\lambda_n(\mu)(\prod_{i=1}^n U_i) > 0$. By Corollary 5.6, we know that there exist a measurable partition $\mathbb{P} = \{B_1, B_2, \ldots, B_r\}$ of X, $D \subset X$ and a positive density subset $\mathcal{A} = \{n_1, n_2, \ldots\}$ of $\mathbb{N}$ such that $D \cap B_i \subset U_i, i = 1, 2, \ldots, n$ and

$$|T^{-n_1}(D \cap \mathbb{P}) \vee T^{-n_2}(D \cap \mathbb{P}) \vee \cdots \vee T^{-n_k}(D \cap \mathbb{P})| \geq 2^{kh}(r-1)^k$$

for each large enough $k \in \mathbb{N}$, where h is a positive constant.

By Lemma 7.1, there exist $b(h) > 0$ and $M_h \in \mathbb{N}$ such that for every natural number $k \geq M_h$ we can find $J_k \subset \{1, 2, \ldots, k\}$ with $|J_k| \geq b(h)k$ and $|\bigvee_{j \in J_k} T^{-n_j}(D \cap \mathbb{P})| = r^{|J_k|}$.

Now, let $\mathcal{R} = \{U_1, U_2, \ldots, U_n\}$. As $D \cap B_i \subset U_i$, $i = 1, 2, \ldots, n$, one gets $|\bigvee_{j \in J_k} T^{-n_j}\mathcal{R}| = n^{|J_k|}$. Set $\delta = \lim_{m \to +\infty} \frac{1}{m}|\{1, 2, \ldots, m\} \cap \mathcal{A}| > 0$. There exists $M \in \mathbb{N}$ such that for any $m \geq M$ we have $\frac{1}{m}|\{1, 2, \ldots, m\} \cap \mathcal{A}| \geq \delta/2$. Set $b = \delta b(h)/2$, $M_b = \max\{M, 2M_h/\delta\}$ and, for each $m \geq M_b$, set $k_m = \max\{j \in \mathbb{N} : n_j \leq m\}$. Then $k_m \geq \frac{\delta}{2} m \geq M_h$. Let $I_m = \{n_j : j \in J_{k_m}\}$. Then one gets

$$I_m \subset \{1, 2, \ldots, m\}, \quad |I_m| = |J_{k_m}| \geq b(h)k_m \geq bm$$

and $|\bigvee_{i \in I_m} T^{-i}\mathcal{R}| = n^{|I_m|}$, i.e., for any $s \in \{1, 2, \ldots, n\}^{I_m}$ there exists $x \in X$ with $x \in \bigcap_{i \in I_m} T^{-i}U_{s(i)}$. Therefore, $U_1, U_2, \ldots, U_n$ have Property P_n.

Conversely, let U_i be a closed mutually disjoint neighborhood of x_i, $i = 1, 2, \ldots, n$. Since U_i is a neighborhood of x_i, $U_1, U_2, \ldots, U_n$ have Property P_n, i.e., there exist $b > 0$ and $M_b \in \mathbb{N}$ such that for each $m \geq M_b$, we can find $J_m \subset \{1, 2, \ldots, m\}$ with $|J_m| \geq bm$, and J_m satisfies that for any $s \in \{1, 2, \ldots, n\}^{J_m}$, there exists $x_s \in \bigcap_{j \in J_m} T^{-j}(U_{s(j)})$. Let $X_S = \{x_s : s \in \{1, 2, \ldots, n\}^{J_m}\}$. Note that for every $t \in \{1, 2, \ldots, n\}^{J_m}$, we have $|\bigcap_{j \in J_m} T^{-j}U_{t(j)}^c \cap X_S| = (n-1)^{|J_m|}$. Combining this fact and $|X_S| = n^{|J_m|}$, one gets

$$N\Big(\bigvee_{j \in J_m} T^{-j}(\mathcal{U})\Big) \geq \Big(\frac{n}{n-1}\Big)^{|J_m|},$$

where $\mathcal{U} = \{U_1^c, U_2^c, \ldots, U_n^c\}$. Thus

$$H\Big(\bigvee_{j \in J_m} T^{-j}\mathcal{U}\Big) \geq |J_m| \log\Big(\frac{n}{n-1}\Big) \geq bm \log\Big(\frac{n}{n-1}\Big).$$

Hence

$$\begin{aligned} h_{\mathrm{top}}(T, \mathcal{U}) &= \lim_{m \to +\infty} \frac{1}{m+1} H(\mathcal{U} \vee T^{-1}\mathcal{U} \vee \cdots \vee T^{-m}\mathcal{U}) \\ &\geq \liminf_{m \to +\infty} \frac{1}{m+1} H\Big(\bigvee_{j \in J_m} T^{-j}\mathcal{U}\Big) \geq b \log\Big(\frac{n}{n-1}\Big) > 0. \end{aligned}$$

This shows that $(x_i)_1^n \in E_n(X, T)$. ∎

An immediate consequence of Theorem 7.3 is

THEOREM 7.4: *Let (X,T) be a topological dynamical system.*

(i) *(X,T) is u.p.e. of order n if and only if (X,T) has Property P_n, where $n \geq 2$.*

(ii) *(X,T) is u.p.e. of all orders if and only if (X,T) has strong Property P.*

For a TDS (X,T), $x \in X$ and a pair of non-empty open subsets U,V of X, let $N(x,U) = \{n \in \mathbb{Z}_+ : T^n(x) \in U\}$ and $N(U,V) = \{n \in \mathbb{Z}_+ : U \cap T^{-n}(V) \neq \emptyset\}$. For $B \subset \mathbb{N}$ and $k \in \mathbb{N}$, we define $B + k = \{b + k : b \in B\}$. Note that two TDSs are weakly disjoint if their product is transitive.

THEOREM 7.5: *If a TDS (X,T) has u.p.e. of order 2, then it is weakly disjoint from any transitive system.*

Proof: Let (Y,S) be a transitive system. Assume that U_1, U_2 and V_1, V_2 are non-empty open subsets of X and Y, respectively. Then

$$N(U_1 \times V_1, U_2 \times V_2) = \{n \in \mathbb{Z}_+ : (T \times S)^{-n}(U_2 \times V_2) \cap (U_1 \times V_1) \neq \emptyset\}.$$

As (Y,S) is transitive, there are $n_0 \in \mathbb{N}$ such that $V = S^{-n_0}(V_2) \cap V_1$ is a non-empty open subset of Y. Thus

$$\begin{aligned} &N(U_1 \times V_1, U_2 \times V_2) \\ &= \{n \in \mathbb{Z}_+ : (T^{-n}(U_2) \cap U_1) \times (S^{-n}(V_2) \cap V_1) \neq \emptyset\} \\ &\supset \{m \in \mathbb{Z}_+ : (T^{-m}(T^{-n_0}(U_2)) \cap U_1) \times (S^{-m}(V) \cap V) \neq \emptyset\} + n_0. \end{aligned}$$

Let $D_0 = U_1$ and $D_1 = T^{-n_0}(U_2)$. Then

$$N(U_1 \times V_1, U_2 \times V_2) \supset N(D_0 \times V, D_1 \times V) + n_0.$$

Now, it remains to show $N(D_0 \times V, D_1 \times V) \neq \emptyset$. As (Y,S) is transitive and V is nonempty, $N(V,V) = N(x,V) - N(x,V)$ contains an IP-set, i.e., there exists a sequence $p_1, p_2, \ldots \in \mathbb{N}$ such that finite sums $p_{i_1} + p_{i_2} + \cdots + p_{i_k} \in N(V,V)$ for each $i_1 < i_2 < \cdots < i_k$, see Theorem 2.17 in [F], where x is a transitive point in V and $A - A = \{a - b \geq 0 : a, b \in A\}$.

As D_0, D_1 have Property P_2, there exist $b > 0$ and $M_b \in \mathbb{N}$ such that for each $m \geq M_b$, we can find $J_m \subset \{1, 2, \ldots, m\}$ with $|J_m| \geq bm$, and J_m satisfies that for any $s \in \{0,1\}^{J_m}$ there exists $x_s \in \bigcap_{j \in J_m} T^{-j}(D_{s(j)})$.

Choose $M \in \mathbb{N}$ with $Mb > 1$ and assume $m > \max\{M_b, (p_1 + p_2 + \cdots + p_M)/b\}$. If $J_m, J_m + p_1, J_m + (p_1 + p_2), \ldots, J_m + (p_1 + p_2 + \cdots + p_M)$ are pairwise disjoint, then

$$|J_m \cup \bigcup_{j=1}^{M} J_m + (p_1 + p_2 + \cdots + p_j)| = (M+1)|J_m| \geq (M+1)bm.$$

Since

$$J_m \cup \bigcup_{j=1}^{M} J_m + (p_1 + p_2 + \cdots + p_j)$$
$$\subset \{1, \ldots, m, m+1, \ldots, m + (p_1 + \cdots + p_M)\},$$

we have $(M+1)bm \le m + (p_1 + p_2 + \cdots + p_M)$. As $bm \ge p_1 + p_2 + \cdots + p_M$ and $Mb > 1$, this is impossible. Thus there exist $1 \le s_1 \le s_2 \le M$ such that $J_m \cap (J_m + \Sigma_{j=s_1}^{s_2} p_j) \ne \emptyset$. Thus, we can find $j_1, j_2 \in J_m$ with $j_1 - j_2 = \Sigma_{j=s_1}^{s_2} p_j$.

Let $s \in \{0,1\}^{J_m}$ with $s(j) = 1$ if $j = j_1$ and 0 otherwise. As $\bigcap_{j \in J_m} T^{-j}(D_{s(j)}) \ne \emptyset$, one gets $T^{-j_1}(D_1) \cap T^{-j_2}(D_0) \ne \emptyset$, that is, $D_0 \cap T^{-(j_1 - j_2)}(D_1) \ne \emptyset$. Thus

$$\sum_{j=s_1}^{s_2} p_j \in N(D_0, D_1) \cap N(V,V) = N(D_0 \times V, D_1 \times V).$$

This ends the proof of Theorem 7.5. ∎

8. Other characterizations and applications

In this section we will give some other characterizations of positive entropy, u.p.e. of order n and all orders. We first show

THEOREM 8.1: *Let (X,T) and (Y,S) be TDS.*

(i) *If (X,T) and (Y,S) have u.p.e. of order n, so does $(X \times Y, T \times S)$, where $n \ge 2$.*
(ii) *If (X,T) and (Y,S) have u.p.e. of all orders, so does $(X \times Y, T \times S)$.*

Proof: (ii) follows from (i) and it remains to show (i). By Theorem 6.4 and Theorem 4.4, there exist $\mu \in M(X,T)$ and $\nu \in M(Y,S)$ such that $\mathrm{Supp}(\lambda_n(\mu)) = X^{(n)}$ and $\mathrm{Supp}(\lambda_n(\nu)) = Y^{(n)}$. Let P_μ (resp. P_ν) be the Pinsker σ-algebra of $(X, \mathcal{B}(X), \mu, T)$ (resp. $(Y, \mathcal{B}(Y), \nu, S)$). It is known that $P_\mu \times P_\nu$ is the Pinsker σ-algebra of $(X \times Y, \mathcal{B}(X \times Y), \mu \times \nu, T \times S)$ (see Theorem 14 of [P]).

For any open sets U_i and V_i of X and Y, $i = 1, 2, \ldots, n$,

$$\begin{aligned}\lambda_n(\mu \times \nu)\Big(\prod_{i=1}^{n} U_i \times V_i\Big) &= \int_{X \times Y} \prod_{i=1}^{n} \mathbb{E}(1_{U_i \times V_i} | P_\mu \times P_\nu) d\mu \times \nu \\ &= \int_X \prod_{i=1}^{n} \mathbb{E}(1_{U_i} | P_\mu) d\mu \cdot \int_Y \prod_{i=1}^{n} \mathbb{E}(1_{V_i} | P_\nu) d\nu \\ &= \lambda_n(\mu)\Big(\prod_{i=1}^{n} U_i\Big) \cdot \lambda_n(\nu)\Big(\prod_{i=1}^{n} V_i\Big) > 0,\end{aligned}$$

since both $\lambda_n(\mu)(\prod_{i=1}^n U_i)$ and $\lambda_n(\nu)(\prod_{i=1}^n V_i)$ are positive. It follows that $\mathrm{Supp}(\lambda_n(\mu \times \nu)) = (X \times Y)^{(n)}$, i.e., $(X \times Y, T \times S)$ has u.p.e. of order n. ∎

We shall say that a subset $\mathcal{B}$ of $\mathbb{N}$ is **interpolating** with respect to subsets $\{U_i\}_{i=1}^n$ if for any $s \in \{1,2,\ldots,n\}^{\mathcal{B}}$ there exists $x_s \in X$ with $x_s \in \bigcap_{j\in\mathcal{B}} T^{-j}U_{s(j)}$.

Theorem 8.2: *Let (X,T) be a TDS and $(x_i)_1^n \notin \Delta_n(X)$ with $n \geq 2$. Then $(x_i)_1^n \in E_n(X,T)$ if and only if for any $\epsilon > 0$ there exists a positive density subset $\mathcal{A}$ of $\mathbb{N}$ such that for any $s \in \{1,2,\ldots,n\}^{\mathcal{A}}$, we can find $x_s \in X$ with $d(T^a x_s, x_{s(a)}) < \epsilon$ for all $a \in \mathcal{A}$.*

Proof: (1) Let $(x_i)_1^n \in E_n(X,T)$. For any $\epsilon > 0$, set

$$U_i = \{x \in X : d(x, x_i) \leq \epsilon/2\} \quad \text{for } i = 1,2,\ldots,n.$$

By Theorem 7.3, $U_1, U_2, \ldots, U_n$ have Property P_n, i.e., there exists $b > 0$ and $M_b \in \mathbb{N}$ such that for any $m \geq M_b$, we can find $J_m \subset \{1,2,\ldots,m\}$ with $|J_m| \geq bm$, and J_m satisfies that for any $s \in \{1,2,\ldots,n\}^{J_m}$ there exists $x_s \in X$ with $x_s \in \bigcap_{j\in J_m} T^{-j}U_{s(j)}$.

Now let

$$\mathcal{F} = \{\mathcal{B} \subset \mathbb{N} : \mathcal{B} \text{ is interpolating with respect to } \{U_i\}_{i=1}^n\}.$$

As $J_m \in \mathcal{F}$ for $m \geq M_b$, $\mathcal{F}$ is nonempty. Set

$$\mathcal{F}(X) = \{y \in \{0,1\}^{\mathbb{N}} : \exists \mathcal{B} \in \mathcal{F} \text{ such that } y(j) = 1 \text{ if and only if } j \in \mathcal{B}\}$$

and let $\sigma: \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ be the shift map.

As $U_1, U_2, \ldots, U_n$ are closed, it is easy to see that $\mathcal{F}(X)$ is closed and σ-invariant, i.e., $(\mathcal{F}(X), \sigma)$ is a subshift. For each $m \geq M_b$, take $y_m \in \mathcal{F}(X)$ with $y_m(j) = 1$ if and only if $j \in J_m$. Assume $\mu_m = \frac{1}{m}\Sigma_{i=0}^{m-1}\delta_{\sigma^i y_m}$ for $m \geq M_b$ and let $\mu = \lim_{i\to+\infty}\mu_{m_i}$ be a limit point of $\{\mu_m\}$ in the weak*-topology. Clearly, μ is a σ-invariant measure.

Note that $\mu([1]) = \lim_{i\to+\infty}\mu_{m_i}([1]) \geq \liminf_{i\to+\infty}(|J_{m_i}|/m_i) \geq b$, where $[1] = \{y \in \mathcal{F}(X) : y(1) = 1\}$. By the ergodic decomposition we know that there exists an ergodic measure ν of $(\mathcal{F}(X), \sigma)$ with $\nu([1]) \geq \mu([1])$. Let $x \in \mathcal{F}(X)$ be a generic point of ν, i.e., $\lim_{m\to+\infty}\frac{1}{m+1}\Sigma_{i=0}^{m-1}\delta_{\sigma^i x} = \nu$. Set $\mathcal{A} = \{j \in \mathbb{N} : x(j) = 1\}$. Then $\mathcal{A}$ is interpolating with respect to $\{U_i\}_{i=1}^n$ and $\mathcal{A}$ is a positive density subset of $\mathbb{N}$, since $\lim_{m\to+\infty}\frac{1}{m}|\mathcal{A} \cap \{1,2,\ldots,m\}| = \lim_{m\to+\infty}\frac{1}{m}\Sigma_{i=0}^{m-1}\delta_{\sigma^i x}([1]) = \nu([1])$.

Now, for any $s \in \{1,2,\ldots,n\}^{\mathcal{A}}$, there exists $x_s \in X$ with $x_s \in \bigcap_{a\in\mathcal{A}} T^{-a}U_{s(a)}$, since $\mathcal{A}$ is interpolating with respect to $\{U_i\}_{i=1}^n$. Thus $T^a x_s \in U_{s(a)}$ for all $a \in \mathcal{A}$. Hence $d(T^a x_s, x_{s(a)}) < \epsilon$ for all $a \in \mathcal{A}$.

(2) Conversely, assume for any $\epsilon > 0$ there exists a positive density subset $\mathcal{A}$ of $\mathbb{N}$ such that for any $s \in \{1,2,\ldots,n\}^{\mathcal{A}}$, we can find $x_s \in X$ with $d(T^a x_s, x_{s(a)}) < \epsilon$ for all $a \in \mathcal{A}$. We shall show that $(x_i)_1^n \in E_n(X,T)$.

For any neighborhood U_i of x_i, take $\epsilon_1 > 0$ with $B_{\epsilon_1}(x_i) \subset U_i$, $i = 1,2,\ldots,n$. Thus there exists a positive density subset $\mathcal{A}$ of $\mathbb{N}$ such that for any $s \in \{1,2,\ldots,n\}^{\mathcal{A}}$, one finds $x_s \in X$ with $d(T^a x_s, x_{s(a)}) < \epsilon_1$ for all $a \in \mathcal{A}$.

Let $b' = \lim_{m\to+\infty} \frac{1}{m}|\mathcal{A} \cap \{1,2,\ldots,m\}|$ and $b = b'/2$. Then $b > 0$ and there exists $M_b > 0$ with $|\mathcal{A} \cap \{1,2,\ldots,m\}| \geq bm$ for all $m \geq M_b$.

Now, for any $m \geq M_b$ set $J_m = \mathcal{A} \cap \{1,2,\ldots,m\}$. Clearly, $|J_m| \geq bm$. For any $s \in \{1,2,\ldots,n\}^{J_m}$, let

$$s'(a) = \begin{cases} s(a) & \text{if } a \in J_m \\ 1 & \text{if } a \notin J_m \end{cases}$$

for all $a \in \mathcal{A}$. For $s' \in \{1,2,\ldots,n\}^{\mathcal{A}}$, there exists $x_{s'} \in X$ with $d(T^a x_{s'}, x_{s'(a)}) < \epsilon_1$ for all $a \in \mathcal{A}$. In particular, for $a \in J_m$ we have $T^a x_{s'} \in B_{\epsilon_1}(x_{s(a)}) \subset U_{s(a)}$. Therefore $x_{s'} \in \bigcap_{j\in J_m} T^{-j}U_{s(a)}$. This shows that $U_1, U_2, \ldots, U_n$ have Property P_n. This implies that $(x_i)_1^n \in E_n(X,T)$. ∎

THEOREM 8.3: *Let (X,T) be a topological dynamical system and $n \geq 2$.*

(i) *(X,T) has u.p.e. of order n if and only if, for any $\epsilon > 0$ and $\{x_1, x_2, \ldots, x_n\} \subset X$, there exists a positive density subset $\mathcal{A}$ of $\mathbb{N}$ such that for any $s \in \{1,2,\ldots,n\}^{\mathcal{A}}$, we can find $x_s \in X$ with $d(T^a x_s, x_{s(a)}) < \epsilon$ for all $a \in \mathcal{A}$.*

(ii) *(X,T) has u.p.e. of all orders if and only if, for any $\epsilon > 0$, there exists a positive density subset $\mathcal{A}$ of $\mathbb{N}$ such that for any $\{x_a\}_{a\in\mathcal{A}} \subset X$, we can find $x \in X$ with $d(T^a x, x_a) < \epsilon$ for all $a \in \mathcal{A}$.*

Proof: Without loss of generality we assume that (X,T) is not trivial. (i) follows from Theorem 8.2. Now we prove (ii). As the sufficiency follows from (i) it remains to show the necessity.

Let (X,T) have u.p.e. of all orders. For any $\epsilon > 0$, there exist $k \geq 2$ and $\{y_i\}_{i=1}^k \subset X$ such that $\bigcup_{i=1}^k B_{\epsilon/2}(y_i) = X$ and $(y_1, y_2, \ldots, y_k) \notin \Delta_k(X)$.

As (X,T) has u.p.e. of order k, $(y_1, y_2, \ldots, y_k) \in E_k(X,T)$. By Theorem 8.2, there exists a positive density subset $\mathcal{A}$ of $\mathbb{N}$ such that for any $s \in \{1,2,\ldots,k\}^{\mathcal{A}}$, we can find $x_s \in X$ with $d(T^a x_s, y_{s(a)}) < \epsilon/2$ for all $a \in \mathcal{A}$.

Now, assume $\{x_a\}_{a\in\mathcal{A}} \subset X$. As $\bigcup_{i=1}^{k} B_{\epsilon/2}(y_i) = X$, there exists $i(a) \in \{1,2,\dots,k\}$ such that $x_a \in B_{\epsilon/2}(y_{i(a)})$ for every $a \in \mathcal{A}$. Let $s \in \{1,2,\dots,k\}^{\mathcal{A}}$ with $s(a) = i(a)$. Then there exists $x_s \in X$ such that $d(T^a x_s, y_{s(a)}) < \epsilon/2$ for all $a \in \mathcal{A}$. As $x_a \in B_{\epsilon/2}(y_{s(a)})$, one has $d(T^a x_s, x_a) \le d(T^a x_s, y_{s(a)}) + d(y_{s(a)}, x_a) < \epsilon$ for all $a \in \mathcal{A}$. ∎

THEOREM 8.4: *Let (X,T) be a topological dynamical system and $n \ge 2$. Then*

(i) *(X,T) has u.p.e. of order n if and only if $(2^X,T)$ has u.p.e. of order n.*

(ii) *(X,T) has u.p.e. of all orders if and only if $(2^X,T)$ has u.p.e. of all orders.*

Proof: It is enough to prove (i). Let d be a metric of X and H_d be the Hausdorff metric of 2^X.

First let $(2^X,T)$ have u.p.e. of order n. For any $\epsilon > 0$ and $\{x_1,x_2,\dots,x_n\} \subset X$, set $A_i = \{x_i\}$. By Theorem 8.3, there exists a positive density subset $\mathcal{A}$ of $\mathbb{N}$ such that for any $s \in \{1,2,\dots,n\}^{\mathcal{A}}$, we can find $A_s \in 2^X$ with $H_d(T^a A_s, A_{s(a)}) < \epsilon$ for all $a \in \mathcal{A}$. For any $x_s \in A_s$, we have $d(T^a x_s, x_{s(a)}) < \epsilon$ for all $a \in \mathcal{A}$. By Theorem 8.2, (X,T) has u.p.e. of order n.

Conversely, let (X,T) have u.p.e. of order n. For any $\epsilon > 0$ and

$$\{A_1,A_2,\dots,A_n\} \subset 2^X,$$

there exists $M \in \mathbb{N}$ such that we can find $B_i = \{x_1^i, x_2^i, \dots, x_M^i\} \subset X$ with $H_d(B_i,A_i) < \epsilon/2$ for $i = 1,2\cdots,n$. By Theorem 8.1, $(X^{(M)},T^{(M)})$ has u.p.e. of order n. Let d_M be a metric of $X^{(M)}$ with $d_M((x_1,x_2,\dots,x_M),(y_1,y_2,\dots,y_M)) = \max_{1\le i\le M} d(x_i,y_i)$.

By Theorem 8.3, there exists a positive density subset $\mathcal{A}$ of $\mathbb{N}$ such that for any $s \in \{1,2,\dots,n\}^{\mathcal{A}}$, we can find $(x_1^s,x_2^s,\dots,x_M^s) \in X^{(M)}$ with

$$d_M((T^{(M)})^a(x_1^s,x_2^s,\dots,x_M^s),(x_1^{s(a)},x_2^{s(a)},\dots,x_M^{s(a)})) < \epsilon/2$$

for all $a \in \mathcal{A}$.

Put $A_s = \{x_1^s,x_2^s,\dots,x_M^s\}$. It is easy to see $H_d(T^a A_s, B_{s(a)}) < \epsilon/2$ for all $a \in \mathcal{A}$. Moreover, $H_d(T^a A_s, A_{s(a)}) < \epsilon$ for all $a \in \mathcal{A}$. This implies that $(2^X,T)$ has u.p.e. of order n by Theorem 8.3. ∎

Now we give a characterization of positive entropy, which was proved by Glasner and Weiss [GW4] when X is a sub-shift and $k = 2$.

THEOREM 8.5: *Let (X,T) be a TDS and $k \ge 2$. Then $h_{\mathrm{top}}(T) > 0$ if and only if there exist k disjoint closed subsets $B_1,B_2,\dots,B_k$ of X and an interpolating set of positive density with respect to $\{B_1,B_2,\dots,B_k\}$.*

Proof: It remains to show the necessity.

Let $h_{\rm top}(T) > 0$. Then $E_k^e(X,T) \neq \emptyset$ by Lemma 6.2. Take $(x_1, x_2, \dots, x_k) \in E_k^e(X,T)$. Since $x_i \neq x_j$ for $1 \leq i < j \leq k$, there exists $\epsilon > 0$ such that $B_i \cap B_j = \emptyset$ for $1 \leq i < j \leq k$, where $B_i = \{x \in X : d(x, x_i) \leq \epsilon\}$ for each $1 \leq i \leq k$. Now by Theorem 8.2, it is easy to see that there exists an interpolating set of positive density with respect to $\{B_1, B_2, \dots, B_k\}$. ∎

9. u.p.e. examples

Let X be a compact metric space and $T: X \to X$ be continuous and surjective. For $n \geq 2$ we may define u.p.e. of order n in the same way. It is easy to see (X,T) has u.p.e. of order n if and only if its natural extension has u.p.e. of order n. Thus in this section we consider continuous surjective maps.

For $p \geq 2$ let $\Lambda = \{0, 1, \dots, p-1\}$ with discrete topology, $\Sigma = \Lambda^{\mathbb{N}}$ with the product topology and $\sigma: \Sigma \to \Sigma$ be the shift. For $n \geq 2$ and $a = (a_1, a_2, \dots, a_n) \in \Lambda^n$ (a block of length n), let $|a| = n$, $\sigma(a) = (a_2, \dots, a_n)$ and $P(a) = (a_2, \dots, a_n, a_1)$. We say a **appears** in $x = (x_1, x_2, \dots) \in \Sigma$ or $x \in \Lambda^m$ with $m \geq n$ if there is $j \in \mathbb{N}$ with $a = (x_j, x_{j+1}, \dots, x_{j+n-1})$ (write $a < x$ for short) and we use t^i to denote $t \cdots t$ (i times). For $b = (b_1, \dots, b_m) \in \Lambda^m$, let $ab = (a_1, \dots, a_n, b_1, \dots, b_m) \in \Lambda^{n+m}$. For $X \subset \Sigma$ and $A \subset X$, let $A^c = X \setminus A$ and

$$[a_1, \dots, a_n] = \{y \in X: (y_1, \dots, y_n) = (a_1, \dots, a_n)\}.$$

For an open cover $\mathcal{U}$ of X let $\mathcal{U}_{i=0}^{n-1} = \mathcal{U} \vee \sigma^{-1}\mathcal{U} \vee \cdots \vee \sigma^{-(n-1)}\mathcal{U}$ and $N(\mathcal{U})$ be the minimal cardinality of subcovers of $\mathcal{U}$. For $K \subset \Lambda^n$ we say K covers X if $\mathcal{U} = \{U_0, U_1, \dots, U_{p-1}\}$ and

$$X = \bigcup_{(i_0, \dots, i_{n-1}) \in K} U_{i_0} \cap \sigma^{-1} U_{i_1} \cap \cdots \cap \sigma^{-(n-1)} U_{i_{n-1}}.$$

Moreover, each $k \in K$ is called a $\mathcal{U}$-name of length n.

Definition 9.1: Let (X,T) be a TDS and U_0, U_1 are two non-empty open subsets of X. We say (X,T) has **Property P** with respect to U_0, U_1 if there is $N > 0$ such that whenever $k \geq 2$, whenever $s = (s(1), \dots, s(k)) \in \{0,1\}^k$, there exists $y \in X$ with $y \in U_{s(1)}, \dots, T^{(k-1)N}(y) \in U_{s(k)}$.

The following lemma is basically Proposition 3 of [B1].

LEMMA 9.2: *Assume that (X,T) has Property P with respect to U_0, U_1 and $U_0 \cap U_1 = \emptyset$. If $\mathcal{R} = \{U, V\}$ is an open cover of X with $U_0 \subset U^c$ and $U_1 \subset V^c$, then $h_{\text{top}}(T, \mathcal{R}) > 0$.*

Proof: As (X,T) has Property P with respect to U_0, U_1, there is $N > 0$ such that whenever $k \geq 2$, whenever $s = (s(1), \ldots, s(k)) \in \{0,1\}^k$, there exists $z(s) \in X$ with

$$z(s) \in U_{s(1)}, \ldots, T^{(k-1)N}(z(s)) \in U_{s(k)}.$$

Thus, if s and s' are two different elements of $\{0,1\}^k$, since $U_0 \subset U^c$ and $U_1 \subset V^c$, the two points $z(s)$ and $z(s')$ cannot be in the same elements of the cover $\mathcal{R}_0^{nN-1}$. Hence $N(\mathcal{R}_0^{nN-1}) \geq 2^n$, and $h_{\text{top}}(T, \mathcal{R}) \geq \frac{1}{N} \log 2$. ∎

LEMMA 9.3: *Let (X,T) be a TDS with a transitive point x. Then T has u.p.e. if and only if for each $i \in \mathbb{N}$, $(x, T^i(x)) \in E_2(X,T)$.*

Proof: Assume that for each $i \in \mathbb{N}$, $(x, T^i(x)) \in E_2(X,T)$. As x is a transitive point we have $\{x\} \times X \subset E_2(X,T)$. Since $E_2(X,T)$ is $T \times T$ invariant we have $\text{Orb}(x) \times X \subset E_X$. Thus $E_2(X,T) = X \times X$. ∎

We will construct a transitive subshift (X, σ) of (Σ, σ) such that $x = (x_1, x_2, \ldots)$ is a transitive point of (X, σ) and satisfies

1. $(x, \sigma^i(x)) \in E_2(X,T)$ for each $i \in \mathbb{N}$,
2. $\mathcal{U} = \{[1]^c, [2]^c, [3]^c\}$ has zero entropy.

More precisely, we will show

THEOREM 9.4: *There is a TDS which is u.p.e. of order 2 and not u.p.e. of order 3.*

Proof: Let $\phi\colon \mathbb{N} \to \mathbb{N}$ such that $(\phi(1), \phi(2), \ldots) = (1,1,2,1,2,3,1,2,3,4,\ldots)$.

Set $A_1 = (00123000)$, $n_1 = |A_1|$ and $\mathcal{U}_1 = \{A_1, \sigma^{\phi(1)}(A_1)0^{\phi(1)}\}$.

Let $C_0^1 = A_1 0^{n_1} = A_{\phi(1)} 0^{n_{\phi(1)}}$ and $C_1^1 = \sigma(A_1)0^1 0^{n_1} \sigma^{\phi^2(1)}(A_{\phi(1)}) 0^{\phi^2(1)} 0^{n_{\phi(1)}}$. Assume

$$\{D_1^1 \cdots D_{n_1}^1, D_{n_1+1}^1 \cdots D_{2n_1}^1, \ldots, D_{n_1 2^{n_1} - n_1 + 1}^1 \cdots D_{n_1 2^{n_1}}^1\} = \{C_0^1, C_1^1\}^{n_1},$$

where $D_j^1 \in \{C_0^1, C_1^1\}$. Set

$$A_2 = A_1 0^{n_1} D_1^1 \cdots D_{n_1}^1 D_{n_1+1}^1 \cdots D_{2n_1}^1 \cdots D_{n_1 2^{n_1} - n_1 + 1}^1 \cdots D_{n_1 2^{n_1}}^1,$$
$$n_2 = |A_2| \quad \text{and} \quad \mathcal{U}_2 = \{A_2, \sigma^{\phi(2)}(A_2) 0^{\phi(2)}\}.$$

If $A_1, \ldots, A_k$ and $\mathcal{U}_1, \ldots, \mathcal{U}_k$ are defined we let

$$C_0^k = A_{\phi(k)} 0^{n_{\phi(k)}} \quad \text{and} \quad C_1^k = \sigma^{\phi^2(k)}(A_{\phi(k)}) 0^{\phi^2(k)} 0^{n_{\phi(k)}}.$$

Assume

$$\{D_1^k \cdots D_{n_k}^k, D_{n_k+1}^k \cdots D_{2n_k}^k, \ldots, D_{n_k 2^{n_k} - n_k + 1}^k \cdots D_{n_k 2^{n_k}}^k\} = \{C_0^k, C_1^k\}^{n_k}.$$

Set

$$A_{k+1} = A_k 0^{n_k} D_1^k \cdots D_{n_k}^k D_{n_k+1}^k \cdots D_{2n_k}^k \cdots D_{n_k 2^{n_k} - n_k + 1}^k \cdots D_{n_k 2^{n_k}}^k,$$
$$n_{k+1} = |A_{k+1}| \quad \text{and} \quad \mathcal{U}_{k+1} = \{A_{k+1}, \sigma^{\phi(k+1)}(A_{k+1}) 0^{\phi(k+1)}\}.$$

It is clear that $n_{k+1} = 2n_k + 2n_{\phi(k)} n_k 2^{n_k} = 2n_k(1 + n_{\phi(k)} 2^{n_k})$.

Let $x = \lim A_k$ and $X = \omega(x, \sigma)$. We claim that (X, σ) is the system we need.

First we show (X, σ) has u.p.e. To do this we need to prove that $(x, \sigma^i(x)) \in E_2(X, T)$ for each $i \in \mathbb{N}$.

Fix $i \in \mathbb{N}$. Suppose that U is a neighborhood of x and V is a neighborhood of $\sigma^i(x)$. By the definition of ϕ, there is k such that $\phi(k) = i$ and $U_0 = [A_k] \subset U$ and $V_0 = [\sigma^i A_k 0^i] \subset V$. Note that $A_k \neq \sigma^i A_k 0^i$. Thus $U_0 \cap V_0 = \emptyset$, and consequently $V_0 \subset U_0^c$ and $U_0 \subset V_0^c$. It is clear that $\mathcal{U}_k = \{A_k, \sigma^i A_k 0^i\}$.

There are infinitely many j such that $\phi(j) = k$. Thus,

$$C_0^j = A_k 0^{n_k} \quad \text{and} \quad C_1^j = \sigma^i A_k 0^i 0^{n_k},$$

and

$$A_{j+1} = A_j 0^{n_j} D_1^j \cdots D_{n_j}^j D_{n_j+1}^j \cdots D_{2n_j}^j \cdots D_{n_j 2^{n_j} - n_j + 1}^j \cdots D_{n_j 2^{n_j}}^j$$

such that $D_{1+ln_j}^j \cdots D_{(l+1)n_j}^j \in \{C_0^j, C_1^j\}^{n_j}$ for $l = 0, 1, \ldots, 2^{n_j} - 1$.

Set $\mathcal{V} = \{U_0^c, V_0^c\}$. It is easy to see that (X, σ) has Property P with respect to U_0, V_0. By Lemma 9.2, we have $h_{\text{top}}(\sigma, \mathcal{V}) > 0$ (let $N = 2n_k$) and consequently $h_{\text{top}}(\sigma, \{U, V\}) > 0$ as $\{U, V\}$ is finer than $\{U_0^c, V_0^c\}$. This proves that $(x, \sigma^i(x)) \in E_2(X, \sigma)$ and hence (X, σ) has u.p.e. according to Lemma 9.3.

Now we show that $\mathcal{U} = \{[1]^c, [2]^c, [3]^c\}$ has zero entropy.

Let $n \in \mathbb{N}$; then $X = \{y \in X : y \in [x_j \cdots x_{j+n-1}], j \in \mathbb{N}\}$ as x is a transitive point. For $\{i_0, \ldots, i_{n-1}\} \in \{1, 2, 3\}^n$ let

$$\alpha(i_0, \ldots, i_{n-1}) = [i_0]^c \cap \sigma^{-1}[i_1]^c \cap \cdots \cap \sigma^{-(n-1)}[i_{n-1}]^c.$$

It is easy to see that $[y_1,\dots,y_n]\subset\alpha(i_0,\dots,i_{n-1})$ if and only if $y_{j+1}\in[i_j]^c$ for $0\le j\le n-1$ and

$$N(\mathcal{U}_0^{n-1})=\min\{|K|\colon K\subset\{1,2,3\}^n \text{ and } \bigcup_{i_0,\dots,i_{n-1}\in K}\alpha(i_0,\dots,i_{n-1})=X\}.$$

As

$$[1]^c=[0]\cup[2]\cup[3],\quad [2]^c=[0]\cup[1]\cup[3],\quad [3]^c=[0]\cup[1]\cup[2],$$

for each $i,j\in\{0,1,2,3\}$ there is $k\in\{1,2,3\}$ such that $[i]\cup[j]\subset[k]^c$. Thus if $a=(a_1,\dots,a_n),b=(b_1,\dots,b_n)\in\{0,1,2,3\}^n$ there is $c=(c_1,\dots,c_n)\in\{1,2,3\}^n$ with

$$[a_1,\dots,a_n]\cup[b_1,\dots,b_n]\subset\alpha(c_1,\dots,c_n).$$

Now fix $k\in\mathbb{N}$. For each $i\le k$, choose $(c_1^i,\dots,c_{n_i}^i)\in\{1,2,3\}^{n_i}$ such that

$$[A_i]\cup[\sigma^{\phi(i)}A_i0^{\phi(i)}]\subset\alpha(c_1^i,\dots,c_{n_i}^i).$$

Then we claim that X is covered by the following $\mathcal{U}$-names of length n_k,

$$\begin{aligned}&P^j(y_1^i,\dots,y_{n_k}^i),\\&(1^jy_1^i,\dots,y_{n_k-j}^i),\\&(y_{j+1}^i,\dots,y_{n_k}^i1^j),\quad 1\le i\le k\quad\text{and}\quad 1\le j\le n_i,\end{aligned}$$

if we write

$$((c_1^i,\dots,c_{n_i}^i1^{n_i})^{n_k/2n_i})=(y_1^i,\dots,y_{n_k}^i),\quad 1\le i\le k-1\quad\text{and}$$
$$(c_1^k,\dots,c_{n_k}^k)=(y_1^k,\dots,y_{n_k}^k).$$

In fact, if a is a block of length n_k and appears in x, then there is $j\ge k$ such that a appears in A_j. Hence by induction on $j\ge k$ it is easy to show the claim. Thus

$$N(\mathcal{U}_{i=0}^{n_k-1})\le\sum_{j=1}^k 4n_k\le 4kn_k\le 4(n_k)^2.$$

This clearly implies that $h_{\mathrm{top}}(\sigma,\mathcal{U})=0$. ∎

A **diagonal system** is one such that $E_2(X,T)$ contains

$$\Delta^1=\{(y,Ty):y\in X\}.$$

Note that u.p.e. implies diagonal. It is shown in [B2] that a diagonal system is disjoint from all minimal systems with zero entropy. Using the idea in the proof of Theorem 9.4, we get a transitive diagonal system without u.p.e., and hence answer a question of [B2] affirmatively. More precisely, we will construct a transitive subshift (X,σ) of (Σ,σ) such that $x = (x_1, x_2, \ldots)$ is a transitive point of (X,σ) and satisfies

1. $(x, \sigma(x)) \in E_2(X,T)$,
2. $(x, \sigma^2(x)) \notin E_2(X,T)$.

If this is the case, then $(\sigma^i(x), \sigma^{i+1}(x)) \in E_2(X,T)$ for each $i \in \mathbb{N}$. As x is a transitive point, for each $y \in X$ one has $(y, \sigma(y)) \in E_2(X,T)$. Moreover, (X,σ) does not have u.p.e., as $(x, \sigma^2(x)) \notin E_2(X,T)$.

THEOREM 9.5: *There is a transitive TDS which is diagonal and does not have u.p.e.*

Proof: Let $\phi\colon \mathbb{N} \to \mathbb{N}$ such that $(\phi(1), \phi(2), \ldots) = (1,1,2,1,2,3,1,2,3,4,\ldots)$.

Set $A_1 = (1020)$, $n_1 = |A_1|$ and $\mathcal{U}_1 = \{A_1, \sigma(A_1)0\}$. Let $C_0^1 = A_1 0^{n_1}$ and $C_1^1 = \sigma(A_1)00^{n_1}$. Assume

$$\{D_1^1 \cdots D_{n_1}^1, D_{n_1+1}^1 \cdots D_{2n_1}^1, \ldots, D_{n_1 2^{n_1} - n_1 + 1}^1 \cdots D_{n_1 2^{n_1}}^1\} = \{C_0^1, C_1^1\}^{n_1}.$$

Set

$$A_2 = A_1 0^{n_1} D_1^1 \cdots D_{n_1}^1 D_{n_1+1}^1 \cdots D_{2n_1}^1 \cdots D_{n_1 2^{n_1} - n_1 + 1}^1 \cdots D_{n_1 2^{n_1}}^1,$$
$$n_2 = |A_2| \quad \text{and} \quad \mathcal{U}_2 = \{A_2, \sigma(A_2)0\}.$$

If $A_1, \ldots, A_k$ and $\mathcal{U}_1, \ldots, \mathcal{U}_k$ are defined, we let

$$C_0^k = A_{\phi(k)} 0^{n_{\phi(k)}} \quad \text{and} \quad C_1^k = \sigma(A_{\phi(k)})00^{n_{\phi(k)}}.$$

Assume

$$\{D_1^k \cdots D_{n_k}^k, D_{n_k+1}^k \cdots D_{2n_k}^k, \ldots, D_{n_k 2^{n_k} - n_k + 1}^k \cdots D_{n_k 2^{n_k}}^k\} = \{C_0^k, C_1^k\}^{n_k}.$$

Set

$$A_{k+1} = A_k 0^{n_k} D_1^k \cdots D_{n_k}^k D_{n_k+1}^k \cdots D_{2n_k}^k \cdots D_{n_k 2^{n_k} - n_k + 1}^k \cdots D_{n_k 2^{n_k}}^k,$$
$$n_{k+1} = |A_{k+1}| \quad \text{and} \quad \mathcal{U}_{k+1} = \{A_{k+1}, \sigma(A_{k+1})0\}.$$

It is clear that $n_{k+1} = 2n_k + 2n_{\phi(k)} n_k 2^{n_k} = 2n_k(1 + n_{\phi(k)} 2^{n_k})$.

Let $x = \lim A_k$ and $X = \omega(x,\sigma) \subset \{0,1,2\}^{\mathbb{Z}_+}$. We claim that (X,σ) is the system we need.

First we will show that $\mathcal{U} = \{[1]^c, [2]^c\}$ has zero entropy, and hence (X, σ) does not have u.p.e. Let $n \in \mathbb{N}$; then $X = \{y \in X: y \in [x_j \cdots x_{j+n-1}], j \in \mathbb{N}\}$ as x is a transitive point. For $\{i_0, \ldots, i_{n-1}\} \in \{1,2\}^n$ let

$$\alpha(i_0, \ldots, i_{n-1}) = [i_0]^c \cap \sigma^{-1}[i_1]^c \cap \cdots \cap \sigma^{-(n-1)}[i_{n-1}]^c.$$

It is easy to see that $[y_1, \ldots, y_n] \subset \alpha(i_0, \ldots, i_{n-1})$ if and only if $y_{j+1} \in [i_j]^c$ for $0 \le j \le n-1$ and

$$N(\mathcal{U}_0^{n-1}) = \min\{|K|: K \subset \{1,2\}^n \text{ and } \bigcup_{i_0,\ldots,i_{n-1} \in K} \alpha(i_0, \ldots, i_{n-1}) = X\}.$$

By the construction of A_i, it is not hard to see that $(12) \not< A_i$ and $(21) \not< A_i$ for any $i \in \mathbb{N}$. As $[1]^c = [0] \cup [2]$, $[2]^c = [0] \cup [1]$, for each $i \in \mathbb{N}$ there exists $(c_1^i, \ldots, c_{n_i}^i) \in \{1,2\}^{n_i}$ such that $[A_i] \cup [\sigma(A_i)0] \subset \alpha(c_1^i, \ldots, c_{n_i}^i)$. Similar to the proof of Theorem 9.4 one has $h_{\text{top}}(\sigma, \mathcal{U}) = 0$.

Now following the proof of Theorem 9.4, we can show that $(x, \sigma(x)) \in E_2(X, T)$. This ends the proof. ∎

It is easy to see that in Theorem 9.4 the set of periodic points in X is dense as for each k and $j \le n_{k+1}/2n_k$, $(A_k 0^{n_k})^j$ appears in x. Thus, there is an invariant measure with full support on X. This of course is also the consequence of the general result: each u.p.e. has an invariant measure with full support [B1].

The following theorem obtained by some modification of the construction of Theorem 9.4 answers a question of [B1, Question 2] negatively. Note that Weiss [W2] has an example which is transitive and has an invariant measure with full support, but there is no ergodic measure with full support.

THEOREM 9.6: *There is a TDS which is u.p.e. and there is no ergodic invariant measure with full support.*

Proof: Instead of A_{k+1} in the construction of Theorem 9.4, we put B_{k+1}. Take $B_1 = (1020)$; we set

$$B_{k+1} = B_k 0^{m_1^k} D_1^k \cdots D_{n_k}^k D_{n_k+1}^k \cdots D_{2n_k}^k \cdots D_{(2^{n_k}-1)n_k+1}^k \cdots D_{2^{n_k} n_k}^k 0^{m_2^k},$$
$$n_{k+1} = |B_{k+1}| \quad \text{and} \quad \mathcal{U}_{k+1} = \{B_{k+1}, \sigma^{\phi(k+1)}(B_{k+1})0^{\phi(k+1)}\},$$

where $m_1^k \ge n_k(n_k^2 2^{n_k} + n_k)$ and $m_2^k \ge n_k m_1^k$.

Set $y = \lim B_k$ and $Y = \omega(y, \sigma)$. (Y, σ) is also a system satisfying Theorem 9.4. Moreover, the set of periodic points of Y is dense in Y and Y has an invariant measure with full support. We now show that there is no ergodic measure on Y with full support.

Assume that μ is an ergodic invariant measure with full support. Let z be a generic point of μ. Then z is also a transitive point of (Y,σ) and we have

$$\frac{1}{n}\sum_{i=0}^{n-1} 1_{[B_1]}\sigma^i(z) \to \int 1_{[B_1]}d\mu = \mu[B_1] > 0,$$

as $1_{[B_1]}$ is a continuous map from Y to $\mathbb{R}$.

Let $N(B_1, C)$ be the number of times that B_1 appears in C. Then there is k_0 such that if $k > k_0$, we have

$$\frac{1}{n_k}\sum_{i=0}^{n_k-1} 1_{[B_1]}\sigma^i(z) \geq \frac{2}{n_{k_0}} \quad \text{and} \quad \frac{N(B_1, B_k)}{n_k} < \frac{1}{n_{k_0}}.$$

By induction, we can show that for $k > k_0$, if $z = (z_1, z_2, \ldots)$ and $C_k = (z_1, \ldots, z_{n_k})$ then C_k only appears in

$$0^{m_1^i} D_1^i \cdots D_{n_i}^i D_{n_i+1}^i \cdots D_{2n_i}^i \cdots D_{(2^{n_i}-1)n_i+1}^i \cdots D_{2^{n_i}n_i}^i 0^{m_2^i},$$

for $i \in \mathbb{N}$ with $\phi(i) \leq k_0$.

This implies that z is not a transitive point, a contradiction. Hence there is no ergodic measure with full support. ∎

Finally, we have

QUESTION 1: *Is there a u.p.e. of order 2 system having an ergodic, even strongly mixing invariant measure with full support but not u.p.e. of all orders?*

QUESTION 2: *Let $(Y, \mathcal{D}, \nu, T)$ be a Lebesgue system and $\mathcal{U}$ be a finite measurable cover of X. Do we have $h_c(T,\mathcal{U}) \geq h_\nu(T,\mathcal{U})$?*

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